Hedging parameter risk

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Abstract

The accurate measurement and effective control of financial risk are of crucial importance to risk managers and regulators. However, risk measures are potentially affected by errors in the estimation of model parameters from limited samples, leading to parameter risk. The key contribution of this paper is the formulation of a general framework to hedge this parameter risk. Applying the new framework to credit portfolio modeling, we highlight the importance of parameter risk, model type, estimation methods, and diversification effects.

Keywords: estimation error, parameter risk, hedging

JEL classification: C13, G13, G32
1 Introduction

The assessment of real-world financial risk (e.g., value-at-risk or expected shortfall) usually involves the estimation of model parameters. In practical applications, the statistical estimates of those parameters are typically treated as if they were the true (i.e., known) values of unknown parameters. As a result, the effects of estimation errors resulting from insufficient statistical data or the incapability to accurately estimate the model parameters are ignored. However, what is optimal in the absence of estimation risk is not necessarily optimal if estimation risk and parameter uncertainty is just neglected (Klein et al., 1978). Thus it is imperative for rational risk managers and prudential regulators to take account of this uncertainty when assessing risk model outcomes.

Pioneering work on the impact of estimation risk on investment decisions and portfolio selection include Kalymon (1971), and Klein and Bawa (1976). Barry and Brown (1985) and Coles and Loewenstein (1988) study effects upon market equilibrium and asset pricing in the presence of uncertain parameters and estimation risk. Common approaches to address estimation risk are Bayesian methods like diffusive priors (Barry, 1974) or informative priors formed via asset pricing models (Pástor, 2000). Other prominent approaches include ‘robust’ estimation principles (Garlappi et al., 2007).

First efforts devoted to the issue of estimation errors on statistical measures of risk exposures in risk management are due to Jorion (1996). Since then, a steadily growing body of literature is emerging. The impact and assessment of parameter and model uncertainty on capital adequacy and risk capital calculations is studied by, e.g., Bao and Ullah (2004), Christoffersen and Gonçalves (2005), Kerkhof et al. (2010), Tarashev (2010), Alexander and Sarabia (2012), Bion-Nadal and Kervarec (2012), Embrechts et al. (2013), Lönnbark (2013), Embrechts et al. (2015), Fröhlich and Weng (2015),

\[ \text{The role of parameter uncertainty in portfolio choice, asset pricing and asset allocation is further investigated in, e.g., Barberis (2000), Veronesi (2000), Xia (2001), Maenhout (2004), and Kan and Zhou (2007). Other studies pertaining to uncertainty in model parameters include option pricing (Bunnin et al., 2002), credit spreads (Korteweg and Polson, 2010), bond portfolios (Feldhütter et al., 2012), and mortgage securitization (Rösch and Scheule, 2014) \textit{inter alia}. The majority of these studies, however, do not focus on risk management issues as such.} \]
and Bignozzi and Tsanakas (2016) among others.² Whereas the meaning of parameter uncertainty in the literature is evident, the notion of model uncertainty is somewhat ambiguous. The uncertainty in the estimates of parameters within a model might also be understood as model uncertainty. Regardless the point of view and even if it were possible to neglect model uncertainty issues due to, e.g., regulatory guidelines, the need to specify unknown parameters within a given model class remains an important issue.³ In essence, the literature provides orientation to add some ‘conservatism’ to estimates from statistical inference or otherwise provision additional capital buffers to cushion for estimation uncertainties.⁴

This paper represents, to the best of our knowledge, the first attempt to formulate a framework that allows financial institutions to hedge parameter risk instead of fully provision for ‘conservative’ parameters. Here, the term hedge is used in its broadest sense, as an offsetting position to potential losses, and parameter risk is the possibility for errors in the parameters. The framework is general, modest in assumptions, and enables contract parties to uniquely determine the fair pricing for parameter risk protection. Such a protection becomes increasingly expedient for circumstances characterized by a high chance for parameter error. This might be particularly pronounced for risk models where estimated risk measures, on average, approach the true, but unknown, risk measure from below, compare, e.g., Figlewski (2004) and Lönnbark (2010). Typically, dependent on the risk measure under consideration, implications become worse with asymmetric distributions and higher tail cutoffs, where a few, but severe, underestimations harm much more than many small ones.

We illustrate our framework applied to credit risk modeling.⁵ Firstly, we conduct a

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²There is also a fast growing literature considering the accuracy and backtesting of risk predictions, see, e.g., Berkowitz and O’Brien (2002), Berkowitz et al. (2009), Escanciano and Olmo (2011), Gouriéroux and Zakoïan (2013), Boucher et al. (2014), and Du and Escanciano (2017) inter alia.

³Therefore the present paper is not concerned with the hedging of model risk as such.

⁴Risk based capital requirements seek to lower the probability of liquidation. However, Hellmann et al. (2000) demonstrate that capital requirements, while putting bank equity at risk and inducing prudent bank behavior, at the same time adversely affect banks’ franchise values, hence stimulating gambling incentives. A similar argument is made by Keppo et al. (2010), who show that higher regulatory capital requirements may postpone recapitalization, thereby actually increasing the institution’s probability of default. Consequently, additional capital burdens to achieve a greater level of safety may negatively affect financial institutions’ behavior.

⁵Estimation errors of parameters are documented to be particularly pronounced in credit risk, Crouhy
simulation study. Employing a common credit portfolio model allowing for closed form hedge premium pricing, we find that (i) a prespecified interval of model parameters can be covered for a small periodic premium payment, (ii) a protection buyer (seller) has an incentive to engage in such a hedge, if the true parameters are underestimated (overestimated), respectively, (iii) the premium is quite robust to the specification of the credit risk model, (iv) a parameter risk hedge to some degree implicitly covers model risk with respect to point estimates from alternative assumptions about the true default generating process, and (v) a protection seller engaging in more than one hedge contract may diversify parameter risk, and decrease its risk of extreme losses. Secondly, an application of our framework to historical one year default rates reveals that the periodic premium of the protection buyer insuring against one weighted standard deviation of the estimates is about 0.02 to 1.66 basis points. Or, subject to contract type and rating grade, about 460 to 1430 years of paying a premium equals the one-off difference between the VaR using the estimates and the VaR for the conservative parameters. We find strong empirical support that in practical applications high rated financial instruments are substantially more prone to parameter errors than lower rated instruments, and—as a result—are more costly in relative terms to be hedged against parameter risk.

For practical applications, parameter risk hedging hinges on institutions willing to sell protection and are capable to steadily provide protection even during periods of high uncertainty surrounding the state of the economy. Prospects could be federal institutions or state funds who themselves are less vulnerable to get into financial distress. In fact, there is ample evidence that financial firms in distress have already been backed by federal agencies, however, without any formal framework, see, e.g., Jorion (2000) and Veronesi and Zingales (2010). The creation of a ‘Parameter Risk Hedge Funds’ administered by, e.g., the major regulatory, treasury, deposit protection et al. (2000) for instance find evidence that industry credit models yield estimates substantially less accurate in comparison to market risk models. Furthermore, Gordy and Heitfield (2010) show that the true model parameters are—on average—underestimated. Other applications of uncertainty in the parameters to credit risk modeling include, e.g., Löffler (2003), Yamai and Yoshiba (2005), Tarashev (2010), and Bernard et al. (2017) among others.
or central bank agencies, might be a suitable candidate to provide protection against
uncertainty with respect to parameter errors. Unlike taxpayer bailouts, such a funds
could mutualize risks among financial firms through an emergency authority. In
addition, a protection seller gets into the unique position to diversify parameter errors
across different parameter risk hedge contracts.

The rest of the paper is organized as follows. Section 2 introduces our framework to
hedge parameter risk. In Section 3, we illustrate the key concepts of our framework,
and discuss real world implications. Section 4 summarizes our findings and concludes.

2 Hedging of parameter risk

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an atomless probability space and \(L^0(\Omega, \mathcal{F}, \mathbb{P})\) a set of \(\mathcal{F}\)-measurable
\(\mathbb{P}\)-a.s. finitely valued random variables on that probability space.

**Definition 1 (Risk of a security).** Let \(M \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})\) be a convex cone, in which any
random variable \(L(Y, \theta) \in M\), with cumulative (discrete) distribution function \(G(\ell, \theta) = \mathbb{P}(L(Y, \theta) \leq \ell), \ell \in [0, 1]\), and probability density (mass) function \(g(\ell, \theta)\), describes a loss of
a risky security (e.g., loan, bond, stock, portfolio). \(Y\) is a random vector and \(\theta\) is a vector
of model parameters. Then \(L(Y, \theta)\) models the risk (e.g., market risk, credit risk) of the
security.

Quantitative tools to aggregate the risk of a security and to express the riskiness
of financial positions in one key figure are risk measures. A risk measure is a function
that maps risk to the real numbers. Therefore any function \(R : M \rightarrow \mathbb{R}\) is a risk
measure. Statistical risk models often require a number of model parameters \(\theta\) to be
specified. In practical applications \(\theta\) is unknown and must be determined, e.g., via
expert judgments, calibration to market data or estimation from past observations.

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\(^6\)Federal or regulatory agencies might already maintain relationships to financial institutions and
closely monitor them. Parameter risk hedge funds do not necessarily introduce additional issues
pertaining to asymmetrical information or excessive risk taking. Discussions on the problems of agency
issues from a regulatory or supervisory perspective include incentives to underestimate VaR (Cuoco and
Liu, 2006), and decrease the quality of risk management systems (Danielsson et al., 2002).
However, the inferred parameters do not necessarily match the true underlying model parameters under a classical statistical perspective.

**Definition 2** (Estimates and parameter error). *Given the parameters according to Definition 1 are unknown, then any specified parameters are estimates \( \hat{\theta} \) and the \((p\text{-norm})\) distance \( \|\theta - \hat{\theta}\|_p \), with real-valued \( p \geq 1 \), is the parameter error.*

**Definition 3** ((Positive/negative) effective parameter error). *For a given risk measure \( R \), an estimate \( \hat{\theta} \) implies an effective parameter error if*

\[
R(L(Y, \theta)) \neq R(L(Y, \hat{\theta})),
\]

*and is called positive effective parameter error if*

\[
R(L(Y, \theta)) - R(L(Y, \hat{\theta})) > 0,
\]

*otherwise it is called negative effective parameter error.*

All estimates lead almost surely to effective parameter errors. In practical applications, a negative effective parameter error—equivalent to an overestimation of the quantified risk of a security—may imply a reduced possibility of a default, which will be costly for a financial institution due to misallocation of capital. On the contrary, a positive effective parameter error—equivalent to an underestimation of the quantified risk of a security—may reduce the cost of capital in the short run, but in the long run it might increase the possibility of a default because the financial institution will not have enough provisions to cover incurred losses.

**Definition 4** (Parameter risk). *The possibility of an effective parameter error is called parameter risk.*

**Remark 1.** *Knight (1921) defines the terms ‘risk’ and ‘uncertainty’ as random variation according to a known or unknown stochastic law, respectively, whereas the type of desired or undesired outcome remains unspecified. In Definition 4 we introduce parameter ‘risk’, since for the definition of a parameter error we assume the model is known with certainty.*
Financial institutions are particularly hurt from positive effective parameter errors, therefore current literature calls to adjust for this particular form of parameter risk by add ons, compare, e.g., Tarashev (2010). However, if these parameters—adjusted for parameter risk—are employed and serve as a basis for the allocation of capital reserves, the provisions may be too costly. Instead of fully providing additional capital reserves, a financial institution may benefit from engaging in a hedge of parameter risk.

**Definition 5** (Payoff of parameter risk hedge). Given a risk measure $\mathcal{R}$, an estimate $\hat{\theta}$ is hedged to the level of $\theta_h$ for

$$\mathcal{R}(L(Y, \hat{\theta})) \leq \mathcal{R}(L(Y, \theta_h)),$$

if a protection seller pays the protection buyer the (possibly discounted) payoff

$$p_c = \begin{cases} 
0 & \text{if } L(Y, \theta) \leq \mathcal{R}(L(Y, \hat{\theta})) \\
L(Y, \theta) - \mathcal{R}(L(Y, \hat{\theta})) & \text{if } \mathcal{R}(L(Y, \hat{\theta})) \leq L(Y, \theta) \leq \mathcal{R}(L(Y, \theta_h)) \\
c (\mathcal{R}(L(Y, \theta_h)) - \mathcal{R}(L(Y, \hat{\theta}))) & \text{if } \mathcal{R}(L(Y, \theta_h)) < L(Y, \theta),
\end{cases}$$

with $c \in \{0, 1\}$.

The contractual parameter $c$ in Definition 5 defines the shape of the payoff structure above the hedge level $\theta_h$ restricted to two polar edge cases. Choosing $c = 1$ the hedge contract can be modeled as a call spread on the underlying state variable $L(Y, \theta)$. Here, a protection seller offers to bear losses beyond a certain threshold $\mathcal{R}(L(Y, \hat{\theta}))$, but subject to a cap $\mathcal{R}(L(Y, \theta_h))$ and, as such, is basically selling mezzanine protection. For $c = 0$ the hedge contract resembles an up and out call option. After breaking the upper barrier nothing is paid to the protection buyer resulting in a much cheaper fee to be paid for this type of protection. Of course, in principle, $c$ could also be modeled more generally within a functional relation to the loss variable $L(Y, \theta_h)$ to allow for more diverse payoff shapes above the hedge level. However, one should keep in mind that values for $c > 1$ would result in an upwards parallel shift of the horizontal leg above
the hedge level with unclear economic interpretation. Likewise, values $c < 0$ seem unreasonable since this would lead to a negative cash flow in case of a breached hedge level. That is, we may interpret $c = 1$ and $c = 0$ as natural lower and upper bounds of protection above the hedge level, respectively.

For both characteristics the compensating payment is restricted by $R(L(Y, \theta_h)) - R(L(Y, \hat{\theta}))$. Thus, by effectively truncating the payoff the protection seller faces no unpredictable and heavy-tailed worst case scenarios.

**Remark 2.** The economic interpretations of the two alternatives for $c$ in Definition 5 should be well considered. A severe economic state described by an extreme realization of $Y$ could imply a large loss. In the case of $c = 1$, the contract will payoff regardless whether parameter risk is causal to the loss. Alternatively, one could argue that a compensation for a possible parameter risk does not seem adequate ($c = 0$) and it is then preferable to liquidate the financial institution.

**Remark 3.** Our framework intends to adjust possibly misspecified parameters upwards to gain additional safety. Alternatively, protection could be provided in the reverse direction. In this case, the attachment point for the resulting payoff profile would be described by $R(L(Y, \theta_h))$ and would be confined to the right by the estimated parameter as an upper level of protection, i.e., $R(L(Y, \hat{\theta}))$. Specifically, cash flows are now tied to a scenario where an involved party has strong belief that their parameter estimates are effectively too conservative and may spare a fraction of some ‘parameter risk buffer’. Though this scenario is conceivable, it seems hard to imagine that it could—in any form—be beneficial for a sound financial system.

Based on these preliminary observations, the fair premium for bearing parameter risk then directly follows via integrating over the loss probabilities given $Y$ and the true model parameters $\theta$. For bearing the parameter risk from the protection buyer, the protection seller receives a fair hedge fee, or, equivalently, a fair hedge premium, defined as the expectation of the payoff, $\mathbb{E}[p_c]$, taken under some appropriate probability measure (see Remark 4).
Theorem 1 (Fair fee $f_c$ for a hedge of parameter risk). The fair fee $f_c = \mathbb{E}[p_c]$ of a parameter risk hedge according to Definition 5 is given by

$$f_0 = f_1 - (D - A)\mathbb{P}(D < L(Y, \theta)), \quad (1)$$

$$f_1 = D - A - \int_A^D \mathbb{P}(L(Y, \theta) \leq \ell)\,d\ell, \quad (2)$$

where $A = \mathbb{R}(L(Y, \hat{\theta}))$ and $D = \mathbb{R}(L(Y, \theta_h))$.

Proof. The payoff $p_1$ corresponds to the difference of a long call with strike $A = \mathbb{R}(L(Y, \hat{\theta}))$ and short call with strike $D = \mathbb{R}(L(Y, \theta_h))$. The contractual payment equals

$$p_1 = (L(Y, \theta) - A)^+ - (L(Y, \theta) - D)^+. \quad (3)$$

Due to the linearity of the expectation operator we simply calculate

$$\mathbb{E}[(L(Y, \theta) - A)^+] = \int_A^1 (\ell - A)\,g(\ell, \theta)\,d\ell = \int_A^1 \ell\,g(\ell, \theta)\,d\ell - A \int_A^1 g(\ell, \theta)\,d\ell$$

$$= \ell G(\ell, \theta)|_A^1 - \int_A^1 G(\ell, \theta)\,d\ell - A G(\ell, \theta)|_A^1$$

$$= 1 - \int_A^1 G(\ell, \theta)\,d\ell = 1 - A - \int_A^1 \mathbb{P}(L(Y, \theta) \leq \ell)\,d\ell,$$

which follows by integration by parts and resembles a call option with strike $A$. Subtraction leads to Equation (2). The payoff $p_0$ equals $p_1$ without $\mathbb{R}(L(Y, \theta_h)) - \mathbb{R}(L(Y, \hat{\theta}))$ if $\mathbb{R}(L(Y, \theta_h)) < L(Y, \theta)$. Due to the linearity of the expectation operator Equation (1) holds.

Remark 4. $\mathbb{E}[p_c]$ in Theorem 1 denotes expectation with respect to at least two probability measures. If it were possible to construct a hedging portfolio through the trading of liquid assets which would perfectly replicate the parameter risk hedge payoff profile, the expectation could be taken under an ‘equivalent martingale measure’. If there is no market absent of arbitrage for the payoff profile under consideration, then ‘risk neutral’ pricing cannot be applied. Thus, the expectation will include some additional risk premium to compensate the protection seller for not being able to fully eliminate all risk. Hence ‘real world’ pricing will
take place under a physical probability measure.

The protection seller pays on average $\mathbb{E}[p_c]$ without necessarily requiring knowledge about the true parameter $\theta$. However, in practical applications $\theta$ is inevitably unknown and fair fees according to Theorem 1 are not computable. For that reason, we introduce a *contractual* fee for a parameter risk hedge.

**Definition 6** (Contractual fee $f^*_c$ for a parameter risk hedge). The contractual fee $f^*_c$ is defined as the expectation of the parameter risk hedge payoff for $L(Y, \theta_h)$ instead of $L(Y, \theta)$. $f^*_c$ accords with $f_c$ in Theorem 1 where $\theta$ is replaced by $\theta_h$.

The Definition 6 implies three advantageous consequences. First, given a hedge range from $\mathcal{R}(L(Y, \hat{\theta}))$ to $\mathcal{R}(L(Y, \theta_h))$ the contractual fee $f^*_c$ is deterministic. Second, the contractual fee $f^*_c$ trivially equals the fair fee $f_c$ if the hedge level $\theta_h$ matches the latent $\theta$. And, third, $\theta_h$ is merely a conservative estimate (or, the result of an add on to an estimate) for $\theta$ and as result the hedge framework does not induce any new parameter risk.

**Remark 5.** A closer look at Equation (2) reveals familiar similarities to ‘classical’ derivatives pricing. Initially considering $\theta$, we obtain the fair fee previously discussed assuming the true parameters are known. In addition, considering $\theta_h$ subject to specified contract terms instead of $\theta$, we arrive at what we label the contractual fee. In conclusion, replacing $\theta$ with $\hat{\theta}$, ultimately yields to conventional call spread pricing.

**Remark 6.** A protection seller gets into the unique position to directly diversify parameter risk across different hedge contracts. If a parameter hedge contract does (not) fulfill the condition

$$\int_{\mathcal{A}}^D \mathbb{P}(L(Y, \theta_h) \leq \ell) d\ell > \int_{\mathcal{A}}^D \mathbb{P}(L(Y, \theta) \leq \ell) d\ell, \quad (3)$$

then it follows from Equation (2) and Definition 6 that the fair fee $f_1$ is greater than (less than or equal to) the contractual fee $f^*_1$ and, as result, the protection seller receives on average less than (more than or equal to) what is paid. This condition, given by the areas
under the cumulative distributions functions ranging from A to D, may be interpreted as an underestimation of the true extreme losses which triggers the payout of a hedge contract.

In real world applications the condition in Equation (3) cannot be verified for a specific hedge contract, since \( \theta \) is essentially unknown. However, if a protection seller engages in more than one hedge contract and the contractual terms therein are not perfectly correlated, then there is a positive likelihood that some contracts will fulfill the condition in Equation (3) and others will not. Thus, there is a chance that the underestimation of the true risk, represented by the condition in Equation (3), is offset—at least to some extent—with an overestimation of the true risk of some other hedge contract. As result, a protection seller is also able to diversify the parameter risk across several protection buyers.

In conclusion, the parameter risk hedge framework, in particular Definition 5 and Theorem 1, is independent from the model type, chosen hedge interval, and holds for any risk measure.

3 Application

3.1 Parameter risk hedge premiums

To begin with, we apply the parameter risk hedge framework to the asymptotic single risk factor (ASRF) model. This model has clear economic interpretations and offers closed form solutions for parameter risk hedge premiums. Underpinning the Basel internal ratings-based (IRB) approach (BCBS, 2006), the ASRF model is well known to academics and practitioners. Gordy and Heitfield (2010) and Tarashev (2010) demonstrate the importance of dealing with parameter risk in this model.\(^7\)

Losses on a homogeneous credit portfolio are assumed to be driven by a systematic risk factor, and each asset represents an infinitesimal share of this portfolio. Idiosyncratic risk disappears owing to full diversification. The distribution of credit portfolio

\(^7\)Section A.1 in the internet appendix further illustrates parameter risk in the ASRF.
loss in a given period is modeled by

\[ L^G(Y, [\rho, \pi]) = \Phi \left( \frac{\Phi^{-1}(\pi) - \sqrt{\rho} Y}{\sqrt{1 - \rho}} \right), \quad Y \text{i.i.d.} \sim \mathcal{N}(0, 1), \]  

(4)

where \( \pi \in (0, 1) \) is an unconditional probability of loan or bond default, \( \rho \in (0, 1) \) is the asset (return) correlation, \( Y \) is a standard normally distributed systematic risk factor, and \( \Phi \) is the standard normal distribution function (with \( \Phi^{-1} \) denoting its inverse).\(^8\)

Given the analytical tractability of the Gaussian ASRF, hedge premiums are straightforward to derive and can be expressed in closed form.

**Corollary 1** (Parameter risk hedge in the Gaussian ASRF). *The Equation (2) from Theorem 1 for the Gaussian ASRF can be easily expressed by the difference of two bivariate standard normal cumulative distribution functions*

\[ \mathbb{E}[p^G_1] = \Phi_2 \left( -\Phi^{-1}(A), \Phi^{-1}(\pi), \rho \right) - \Phi_2 \left( -\Phi^{-1}(D), \Phi^{-1}(\pi), \rho \right), \]  

(5)

with

\[ \rho = -\sqrt{1 - \rho}, \quad A = \mathcal{R}(L^G(Y, [\hat{\rho}, \hat{\pi}])), \quad \text{and} \quad D = \mathcal{R}(L^G(Y, [\rho_h, \pi_h])), \]

where \( \Phi_2(\cdot, \cdot; \rho) \) denotes the bivariate standard normal cumulative distribution function with correlation parameter \( \rho \).

Additionally \( \mathbb{P}(D \leq L^G(Y, [\rho, \pi])) \) in Equation (1) can be expressed in closed form, leading to

\[ \mathbb{E}[p^G_0] = \mathbb{E}[p^G_1] - (D - A) \Phi \left( \rho \Phi^{-1}(D) + \Phi^{-1}(\pi) \right). \]  

(6)

**Proof.** Using Equation (30c) in Andersen and Sidenius (2005). \( \square \)

In the sequel, we consider the common value-at-risk (VaR) and conditional value-at-
risk (CVaR) measures. For the Gaussian ASRF these read

$$\text{VaR}^G_{\alpha}(\rho, \pi) = \Phi \left( \frac{\Phi^{-1}(\pi) - \sqrt{\rho} \Phi^{-1}(1 - \alpha)}{\sqrt{1 - \rho}} \right),$$

and

$$\text{CVaR}^G_{\alpha}(\rho, \pi) = \frac{1}{1 - \alpha} \Phi \left( \frac{\Phi^{-1}(\pi), \Phi^{-1}(1 - \alpha), \sqrt{\rho}}{\sqrt{1 - \rho}} \right).$$

Next, we illustrate general properties and magnitudes of hedge premiums for different contract types and risk measures. During our case studies, we focus on the essentials of the parameter risk hedge framework and refrain from discussing possible influences due to risk preferences of economic agents. That is, we consider zero risk premiums and undiscounted payoffs.

For the asset correlation, we assume $\rho = 20\%$, which lies in between the lower and upper bounds found in the Basel Accords. The probabilities of default are $\pi \in \{0.2\%, 1\%, 5\%\}$, roughly corresponding to historical one year default rates of Moody’s Baa, Ba, and B rated corporates (compare Table 4). We further assume that the estimation leads to an underestimation of 25% for each parameter.\(^9\) Since we know the true parameters, we can calculate the fair hedge fee, when the true parameter is hedged, i.e., the hedge range is defined from $\hat{\theta}$ to $\theta_h = \theta$.

Table 1 provides the results for three different parameter settings and for the risk measures VaR and CVaR for the confidence levels $\alpha = 99\%$ and $\alpha = 99.9\%$. The expected hedge premiums for the barrier option like payoff structure $f_0$ is less than the corresponding expected call spread option payoff $f_1$, which, from Equation (1), is by construction. Otherwise, the two payoff types share similar behaviors, thus we will focus on the case $c = 0$ for further discussions.

The resulting fair hedge premiums range from 0.17 to 16.73 basis points.\(^10\) From the analysis presented in Figure A.1 of the internet appendix, we learn that such an underestimation for the shortest time horizon with $T = 7$ years and lowest probability of default occurs in around 30% of the cases.

In comparison, e.g., the FDIC charges total base assessment rates around 7–77.5 basis points annually for all risk categories, compare FDIC (2011).
Table 1: Comparison of fair hedge premiums and risk measures for different parameter settings, contractual types, and confidence levels. Entries report the resulting risk measures $R$, i.e., VaR and CVaR, alongside with their corresponding fair hedge premiums $f_c = E[p_c]$, i.e., the hedge parameter $\theta_h$ equals $\theta$, for two contract types $c \in \{0, 1\}$, confidence levels $\alpha \in \{99\%, 99.9\%\}$ and the hedge range is from $\hat{\theta}$ to $\theta_h = \theta$. For the Gaussian ASRF model the relevant vector of model parameter is $\theta = [\rho, \pi]$. The first two columns display the underlying true parameter vectors $\theta$, while the next two columns represent possible estimates of the parameters $\hat{\theta}$ with an assumed 25% underestimation of the true model parameters.

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<tr>
<th>True [%]</th>
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<th>Risk measures / Premiums</th>
<th>VaR</th>
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instance, for $\alpha = 99.9\%$ and the medium default risk bucket ($\pi = 1\%$) an estimated VaR of 9.01% is hedged to the level of 14.553%. For this hedge level the protection buyer has to pay a fee of 0.929 bp, which is equivalent to $596 \approx \frac{14.553\% - 9.01\%}{0.929\text{bp}}$ years of paying a fee instead of increasing capital for achieving the same degree of safety.

Given the parameter constellations in Table 1 the fair hedge premiums decrease with higher $\alpha$. While the true VaR roughly multiply with 2.5 ($\pi = 0.2\%$), 2 ($\pi = 1\%$), and 1.5 ($\pi = 5\%$) by increasing $\alpha$ from 99% to 99.9%, the fair hedge premiums (in absolute terms) decrease by 0.60, 0.45 and 0.30, respectively. This behavior is expected since with $\alpha$ getting larger, actual payoffs from the parameter risk hedge become less probable.

By definition, the VaR are less than CVaR for all confidence levels, however, the fair hedge premiums for CVaR are consistently lower than they are for the VaR. Since the CVaR is equal to VaR with a higher confidence level, the smaller hedge fee for the CVaR follows directly from the results for a higher $\alpha$. 
For the same parameter error (in all cases each parameter is underestimated by 25%) the risk buckets with lowest probability of default react more sensitive. The relative error is 60% (55% for $\pi = 1\%$, 45% for $\pi = 5\%$). Therefore the lower risk buckets are apparently more prone to parameter errors. However, the relative premium $f_c/R(\theta)$ is with 0.213%, 0.283%, 0.352% smaller, respectively. Thus, although higher rated risk buckets are more prone to parameter errors, the true hedge fee is (relatively) smaller for the same relative magnitude of parameter errors.

Next, we analyze how the fair hedge premiums $f_c$ relate to contractual fees $f_c^\ast$. These premiums are deterministic with respect to a chosen hedge range, however, the hedge range is stochastic because it depends on the random realization of the estimates $\hat{\theta}$ and standard errors $\hat{\sigma}(\hat{\theta})$. For each case, we sample credit losses according to Equation (4). During each iteration, we employ the estimation method described in Duellmann et al. (2010) providing analytical solutions for the estimates $\hat{\theta} = [\hat{\rho}, \hat{\pi}]$ with corresponding standard errors $\hat{\sigma}(\hat{\theta})$. For this illustration, we define the hedge parameter as the sum of the estimate and its standard error weighted by a factor $\kappa = 75\%$. Therefore, for each single realization, we obtain different fair and contractual hedge premiums.

Figure 1 shows the realization of the distributions for the VaR sorted in ascending order, and the corresponding fair fees and contractual fees for two contract types $c \in \{0, 1\}$. The true parameters are $[\rho, \pi] = [20\%, 1\%]$ and the time horizons are 15 and 30 years.

In the left panel, we plot the distributions of the hedge range given by the difference of the lower attachment point $A$ fixed by the VaR at the estimates as well as the upper detachment point $D$ confined by the VaR at the hedged parameter at confidence level $\alpha = 99.9\%$. The vertical, dashed gray lines indicate the intersections where the estimated and hedged VaR equal the true VaR. These crossing points separate the simulated distributions in each subplot into three different areas labeled (I), (II), and (III).

(I) $A \leq D \leq \text{VaR}(\theta)$. From a hedging perspective, the product is beneficial, since the true risk is clearly underestimated (intersection from $D$ to $\text{VaR}(\theta)$).
Figure 1: Effects of parameter risk on hedge premiums. The left figure shows the distributions of $A$ given by $\text{VaR}(\hat{\theta})$ (solid lines) and $D$ given by $\text{VaR}(\theta_h)$ (dashed lines) in comparison to the true $\text{VaR}(\theta)$ (horizontal lines) at confidence level $\alpha = 99.9\%$ for $\theta = [\rho, \pi]$ with $\rho = 20\%$ and $\pi = 1\%$. The right figure shows the contractual fee $f^*_c$ in comparison to the resulting fair fee $f_c$ (i.e., the average expected payoff from the product) for the two contract types $c = 0$ (gray) and $c = 1$ (black). The upper two figures are for $T = 15$, and the lower two figures are for $T = 30$. The vertical, dashed gray lines indicate the intersecting points where the estimated and hedged VaR equal the true VaR. The simulation is repeated $10^4$ times and results binned into 20 equally spaced intervals. The graphs are sorted by $A$, all other data points are adjusted accordingly. The solid and dashed lines plot the mean values, the dotted lines depict the corresponding 10% and 90% quantiles, respectively.
(II) $A \leq \text{VaR}(\theta) \leq D$. This case describes a situation where the true risk is—on average—slightly overhedged. Without the hedge product, however, the true risk would remain underestimated.

(III) $\text{VaR}(\theta) \leq A \leq D$. The true risk is overestimated.

In the right panel, we plot the contractual fee $f_c^*$ in comparison to the resulting fair fee $f_c$ for the two contractual types $c \in \{0, 1\}$. The passage from case (I) to (II) occurs, if $f_c = f_c^*$. That is, in this particular case, when the upper hedge range $D$ correspond to the true VaR, then the hedge premium—irrespective of the contractual type—naturally equals the fair fee.

(I) The fair fee (expected payoff) is greater than the contractual fee. Recalling from the left panel that a parameter hedge is necessary, the protection buyer profits in two respects. Protection is needed, and the net present value for the protection buyer is positive.

(II) A parameter hedge is necessary, however, this comes at the cost of a slight overprotection. Thus, the net present value is negative.

(III) Any unit of extra protection will disproportionately drive associated cost.

The three cases each make up roughly one third of the relative frequency of occurrences. With $T$ getting smaller, we observe a right shift of these divisions. This shift implies, that for higher parameter risk a hedge becomes increasingly reasonable for the protection buyer.

For the contractual type $c = 1$, the product is more vulnerable to parameter misspecifications viewed from both sides of protection, which becomes evident through the much steeper gradients in the right panel in comparison to $c = 0$.

Given the generality of our framework, the same kind of analysis conducted here could be applied to any model and risk type. With such a procedure the sensitivity of a specific model to parameter risk becomes assessable.
3.2 Diversification of parameter risk

As demonstrated by the analysis presented in Figure 1, the product translates the realized parameter risk of an underlying portfolio to a positive or negative difference of the fair fee $f_c$ to the contractual fee $f^*_c$. Moreover, as argued at the end of Section 2, a protection seller may diversify this risk by engaging in more than one hedge contract with different protection buyers. A requirement for such diversification is that the hedge contracts are at least to some extent independent of each other.

We extend the ASRF framework following Pykhtin and Dev (2002) to demonstrate this diversification effect. The systematic single risk factor $Y$ in Equation (4) now turns into a sectoral risk factor. Thus $Y^j$ is composed of a super-systematic factor $Y^*$, representing the overall macro-economy, and a sector $j$ specific risk factor $U^j$, according to

$$Y^j = \sqrt{\delta} Y^* + \sqrt{1 - \delta} U^j,$$

where $Y^*$ and $U^j$ are i.i.d. standard normal. This extension allows modeling credit risky portfolios covering different sectors and the linear dependence structure between the sectors is explicitly described by $\delta \in [0,1]$. If $\delta$ equals one, all sectors are perfectly correlated and are only driven by the super-systematic factor, while if $\delta$ equals zero, the super-systematic factor has no impact on the firms and they are only jointly affected by the sector-specific risk factor and become independent from each other.

Next, we assume that the protection seller engages in two hedge contracts, covering two credit risky portfolios. These portfolios only differ in sectors, all other settings being equal. The true parameters of each portfolio is $\rho = 20\%$ and $\pi = 1\%$. However, again these parameters are unknown and have to be estimated based on observable losses for $T = 15$ years using the analytical estimation method outlined in Duellmann et al. (2010). For both contracts, the hedge parameter is the sum of the estimated parameters and their standard errors weighted by $\kappa = 75\%$ and the VaR is calculated at confidence level $\alpha = 99.9\%$. For each sample, we calculate the contractual and fair
fee for each contract. The protection seller receives \( f^*_c = f^{*1}_c + f^{*2}_c \) and has to pay on average \( f_c = f^1_c + f^2_c \), where \( f^{*i}_c \) and \( f^i_c \) is the contractual and fair fee of hedge contract \( i \in \{1, 2\} \). We repeat this sampling procedure \( 10^4 \) times for \( \delta \in \{0\%,50\%,75\%,100\%\} \) and Figure 2 summarizes the resulting fees with similar interpretation as in Figure 1.

![Graph showing contractual fees compared to resulting fees](image)

**Figure 2: Diversification of parameter risk.** This figure shows the contractual fee \( f^*_c = f^{*1}_c + f^{*2}_c \) in comparison to the resulting fair fee \( f_c = f^1_c + f^2_c \) (i.e., the average expected payoff from the product) for the two contract types \( c = 0 \) (left graph) and \( c = 1 \) (right graph), for a protection seller engaging in two hedge contracts. \( f^{*i}_c \) (\( f^i_c \)) is the contractual (fair) fee of hedge contract \( i \in \{1, 2\} \). Both underlying portfolios differ sector wise and the dependence between these sectors is modeled by \( \delta \), a larger value of \( \delta \) indicates a higher correlation and both sectors are perfectly (un)correlated for \( \delta = 1 \) (\( \delta = 0 \)). All other parameters are assumed to be equal, i.e., \( \Theta = [\rho, \pi] \) with \( \rho = 20\% \) and \( \pi = 1\% \). The simulation is repeated \( 10^4 \) times and results binned into 20 equally spaced intervals. The graphs are sorted by \( A \) (not depicted), all other data points are adjusted accordingly.

If \( \delta \) equals one, both portfolios are solely driven by the super-systematic factor and result in the same default rates. Hence, all estimated parameters and hedge parameters are the same, leading to exactly the same fees. Therefore the figure equals the graph presented in the upper-right hand graph in Figure 1, with all values simply doubled. However, with \( \delta < 1 \) the resulting default rates increasingly differ and, as a result, the corresponding hedge fees.

To quantify and compare potential diversification benefits for different degrees of \( \delta \), we consider perfect correlation with \( \delta = 1 \) as a reference case (i.e., no diversification). The differences from the fair fees (dashed lines) to the corresponding contractual fees (solid lines) depicts the actual parameter error, i.e., the higher the difference, the higher the error. To aggregate the errors over the relative frequency of occurrences, we calculate the (absolute) areas \( A_\delta \) confined by these two lines. Next, we compare the
resulting areas from the remaining $\delta$ and relate them to the base case given by $\delta = 1$. The results of the diversification benefit $D_c = \frac{A(\delta<1)-A(\delta=1)}{A(\delta=1)}$ are summarized in Table 2.

Table 2: Diversification benefits. This table reports beneficial diversification effects in case of imperfect correlation from a super-systematic factor to a sectoral risk component. The case $\delta = 1$, i.e., perfect correlation, serves as the base case in relation to values $\delta < 1$. The diversification benefit $D_c$ is given by $\frac{A(\delta<1)-A(\delta=1)}{A(\delta=1)}$, where $A$ is the (absolute) area between the fair and contractual fees for two contract types $c = 0, 1$. The computations of $D_c$ are based on the simulation results summarized in Figure 2.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$D_0$ [%]</th>
<th>$D_1$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>-11.38</td>
<td>-11.37</td>
</tr>
<tr>
<td>0.50</td>
<td>-25.73</td>
<td>-28.56</td>
</tr>
<tr>
<td>0.00</td>
<td>-42.87</td>
<td>-49.03</td>
</tr>
</tbody>
</table>

All six cases exhibit the potential for diversification effects. Even for a relatively high correlation with $\delta = 0.75$, the average (absolute) deviances from $f_c$ to $f_c^*$ diminish by more than 11% for both contract types. This gets more pronounced with decreasing values for $\delta$ up to the edge case with $\delta = 0$, where we observe a reduction of premium differences of nearly 50%. Furthermore, for contract type $c = 1$ and the scenario under consideration, there seems to be more potential for diversification effects in comparison to $c = 0$ with decreasing $\delta$. This is important given that the fee is for $c = 1$ more sensitive to parameter misspecification as discussed above.

In conclusion, we see with lower correlation between sectors that the difference of the fair fee and contractual fees decreases. As a result the protection seller can diversify the parameter risk across two protection buyers. Some degree of independence of the premium contracts may additionally emerge, if the protection buyer employs different modeling approaches, differ in implementation details or have differing data quality.

### 3.3 Model sensitivity

One advantage of our parameter risk hedge framework is the applicability of any risk model with tractable loss distributions. To analyze alternative distributional assumptions, we consider generalized ASRF specifications by using more sophisticated distributions for the factors. Empirical asset pricing literature has documented the
stylized fact that asset returns possess heavier tails than predicted by the normal distribution (Cont, 2001). Therefore, we compare the Gaussian to the Student-$t$ and the normal inverse Gaussian (NIG) copula model in the framework of homogeneous portfolios. It is advantageous that in all cases, the main model parameters are the same, i.e., the asset (return) correlation $\rho$ and the probability of default $\pi$, and share the same interpretation. Besides these commonalities, we treat $\nu$ and $\alpha$ representing the heavy-tailedness of the Student-$t$ and NIG distribution as hyperparameters.\(^{11}\) For both of these loss distributions, the Gaussian occurs as a limit case, letting $\alpha, \nu \to \infty$.

For the Student-$t$ copula model the credit portfolio loss in a given period is modeled by
\[
L^T(X, Y, [\rho, \pi]) = \Phi \left( \frac{\sqrt{\frac{2}{\nu}} T_{\nu}^{-1}(\pi) - \sqrt{\rho} Y}{\sqrt{1 - \rho}} \right), \quad X \overset{\text{i.i.d.}}{\sim} \chi^2_{\nu}, \quad Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1),
\]
where $\chi^2_{\nu}$ is the chi-squared distribution with $\nu$ degrees of freedom and $T_{\nu}^{-1}$ is the inverse of the Student-$t$ distribution, see, e.g., Hamerle and Rösch (2005).

For the NIG copula model the credit portfolio loss in a given period is modeled by
\[
L^{\text{NIG}}(M, [\rho, \pi]) = F_{\text{NIG}} \left( \frac{E^{-1}_{\text{NIG}} \left( \frac{1}{\nu} \right) (\pi) - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right), \quad M \overset{\text{i.i.d.}}{\sim} F_{\text{NIG}(1)},
\]
where following the notation of Kalemanova et al. (2007) $F_{\text{NIG}(s)} := F_{\text{NIG}}(x; s\alpha, s\beta, -s\frac{\beta^2}{\alpha^2}, s\frac{\gamma^2}{\alpha^2})$ is the normal inverse Gaussian distribution function. To reduce the number of distribution parameters, $\beta$ is set to zero which makes the distribution symmetric and setting $\gamma = \sqrt{\alpha^2 - \beta^2}$ makes the distribution having zero mean and unit variance.

To investigate the sensitivity of fees to model choice, we now look at a case with fixed hedge levels. The hedge range can be defined without specification of a model

\(^{11}\) Note, that in the Gaussian copula model, the location and scale parameter of a normal distribution are set to have mean zero and unit variance. Hence, these parameters are also hyperparameters. Standardizing the distributions, and thus reducing the number of parameters to be fitted, is not only convenient but also necessary to allow for the interpretation of $\rho$ as a correlation coefficient in valid statistical terms.
and corresponding parameters. Fees, on the other hand, depend on a model with the corresponding model parameters. Therefore we can calibrate model parameters to a given hedge range. To explore this relation of fair premiums to our models under consideration, we recall the results from Table 1 for the Gaussian model and analyze the case given by $\rho = 20\%$ and $\pi = 1\%$ to be consistent with previous sections. With these parameters, the VaR for the Gaussian benchmark yields to $\text{VaR} = 14.5525\%$ at the $\alpha = 99.9\%$ confidence level. The estimates, assumed to be again underestimated by 25%, are set to $\hat{\rho} = 15\%$ and $\hat{\pi} = 0.75\%$ yielding $\text{VaR}(\hat{\theta}) = 9.0102\%$. This will fix our hedge range and serves as a benchmark to which we will calibrate the other two models. The Student-$t$ and NIG model both have three parameters. By the construction of the ASRF, the probability of default $\pi$ equals the expectation of the three distributions under consideration. This allows us, given $\pi$ and VaR to back out precisely the biunique $\rho_{\text{impl}}$ implied by the hedge range from the Gaussian benchmark and given a degree of heavy-tailedness $\nu(\alpha)$ for the Student-$t$ (NIG) model, respectively. Hence, for a given set of tail measures for the two model alternatives, we infer $\rho_{\text{impl}}$ by solving

$$
\text{VaR}_\alpha^G(\rho, \pi) = \text{VaR}_\alpha^T(\rho_{\text{impl}}, \pi, \nu) = \text{VaR}_\alpha^{\text{NIG}}(\rho_{\text{impl}}, \pi, \alpha) = 14.5525\%,$$

and calculate the corresponding fees $f_c$ for the Student-$t$ and NIG model. We emphasize that for pricing purposes, we only need the hedge parameter and the hedge range from $9.0102\%$ to $14.5525\%$ provided by the Gaussian model.

The range of tail measures we consider is guided by reported values from the literature and own estimations based on physical loss data. For instance, Hull and White (2004) report four degrees of freedom showing a good fit to iTraxx tranche data and Kalemanova et al. (2007) find a $\alpha = 0.4794$ calibrated to iTraxx tranches. These represent ‘risk neutral’ parameters implied from market expectations which are known to describe much heavier tails in the return distributions than what is typically observed for physical return distributions. Based on rating performance data from Standard & Poor’s, Hamerle and Rösch (2005) find approximately 33 degrees of freedom for an application to the rating grade BB, which confirms that ‘real world’ asset return
distributions inferred from historical default rates are not as heavy-tailed as typically result from calibration to market data.

Fitting the two models under consideration to historical one year default rates, we find that for the Student-$t$ model the estimated degrees of freedom $\nu$ are about 25, while for the NIG shape parameter $\alpha$, we find a value about 5. Furthermore, it turns out that the fitted $\pi$ are quite stable and hardly distinguishable from each other, which provides empirical support holding $\pi$ fixed according to the value obtained via the Gaussian model. Moreover, we find that, in general, the VaR($\theta$) from the Gaussian model cover all VaR($\hat{\theta}$) of the other models with more sophisticated factor distributions. Thus, a parameter risk hedge with a Gaussian hedge implicitly insures against model risk with respect to point estimates from the alternative model specifications.

In Table 3, we report the implied correlations $\rho_{impl}$ as well as the resulting model based premiums $f_c$ for the prespecified hedge range. The last two columns report the price ratio $Q_c = \frac{f_c - f^G_c}{f^G_c}$ of the Student-$t$ or NIG model to the Gaussian case (denoted by $G$) for both contractual types $c \in \{0, 1\}$. For the Student-$t$ and the NIG model we consider seven different tail measures each. These range from typical parameter values calibrated to market expectations, over parameters typically estimated from historical loss data, and finally to parameter values approaching the Gaussian limit, letting $\alpha, \nu \rightarrow \infty$.

The implied $\rho_{impl}$ shown in Table 3 are decreasing relative to the Gaussian benchmark case if the shape parameter for both alternatives, the Student-$t$ and NIG models, are getting smaller leading to heavier tails than the normal. This indicates that the shape parameter and the parameter $\rho$ are interrelated since they both influence the skewness of the loss distributions.

Comparing the relative changes $Q_c$ to the Gaussian model, we see different signs for the two model alternatives. The heavier the tails for a Student-$t$ copula model, fair premiums are increasing. We find the reverse for the NIG model. Here, the heavier

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12 The Student-$t$ and NIG model parameters are estimated with a correlated binomial model instead of the ASRF, compare Section 3.4 for the data and methodology. Estimation results for the model alternatives are reported in Section A.3 of the internet appendix.

13 For $\nu = 5$, we observe a departure from this pattern. The change of sign in this particular case is
Table 3: Model sensitivity of fair fees. This table reports correlation parameters $\rho_{impl}$ for the Student-$t$ and NIG model implied from a Gaussian benchmark model (denoted by $G$) given by $\pi = 1\%$, $\rho = 20\%$, and $\text{VaR}_{\alpha=99.9\%}(\rho, \pi) = 14.5525\%$. Since the expected value equals the probability of default in all three models, the $\rho_{impl}$ are the biunique outcomes for given $\pi$ and VaR for seven prespecified degrees of heavy tailedness encoded by $\nu$ and $\gamma$. The fourth and fifth column report the fair premiums for two contractual types $c \in \{0, 1\}$ in basis points. The last two columns report the relative difference to the benchmark Gaussian fair fees $Q_c = \frac{f^G_c - f^c}{f^G_c}$. The entries are grouped into three panels. The upper panel reports the values from the Gaussian benchmark model. The lower two panels each report the results of a Student-$t$ and NIG copula model for different sets of heavy tailedness, respectively.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\nu = \alpha$</th>
<th>$\pi [%]$</th>
<th>$\text{VaR}_{\alpha} [%]$</th>
<th>$f_0 [\text{bp}]$</th>
<th>$f_1 [\text{bp}]$</th>
<th>$Q_0 [%]$</th>
<th>$Q_1 [%]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss</td>
<td>$\nu = \alpha$</td>
<td>$\pi = 1$</td>
<td>14.5525</td>
<td>0.9294</td>
<td>1.4837</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\nu = \alpha$</td>
<td>$\pi = 1$</td>
<td>14.5525</td>
<td>0.9307</td>
<td>1.4850</td>
<td>0.1390</td>
<td>0.0871</td>
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<tr>
<td>Student-$t$</td>
<td>$\nu = 100$</td>
<td>14.5525</td>
<td>0.9364</td>
<td>1.4907</td>
<td>0.7516</td>
<td>0.4708</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu = 50$</td>
<td>14.5525</td>
<td>0.9606</td>
<td>1.5148</td>
<td>3.3495</td>
<td>2.0983</td>
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<tr>
<td></td>
<td>$\nu = 25$</td>
<td>14.5525</td>
<td>0.9800</td>
<td>1.5343</td>
<td>5.4429</td>
<td>3.4097</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu = 20$</td>
<td>14.5525</td>
<td>1.0262</td>
<td>1.5804</td>
<td>10.4085</td>
<td>6.5203</td>
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</tr>
<tr>
<td></td>
<td>$\nu = 15$</td>
<td>14.5525</td>
<td>1.0931</td>
<td>1.6583</td>
<td>16.6085</td>
<td>10.4497</td>
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</tr>
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<td></td>
<td>$\nu = 10$</td>
<td>14.5525</td>
<td>1.1956</td>
<td>1.7498</td>
<td>28.6358</td>
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</tr>
<tr>
<td></td>
<td>$\nu = 5$</td>
<td>14.5525</td>
<td>0.4750</td>
<td>1.0293</td>
<td>48.8920</td>
<td>30.6283</td>
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<tr>
<td>NIG</td>
<td>$\gamma = 10$</td>
<td>14.5525</td>
<td>0.9444</td>
<td>1.4587</td>
<td>2.6909</td>
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<tr>
<td>$\gamma = 5$</td>
<td>14.5525</td>
<td>0.8411</td>
<td>1.3954</td>
<td>9.5008</td>
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<tr>
<td>$\gamma = 4$</td>
<td>14.5525</td>
<td>0.8025</td>
<td>1.3568</td>
<td>13.6541</td>
<td>8.5536</td>
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<tr>
<td>$\gamma = 3$</td>
<td>14.5525</td>
<td>0.7369</td>
<td>1.2911</td>
<td>20.7199</td>
<td>12.9799</td>
<td></td>
<td></td>
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<tr>
<td>$\gamma = 2$</td>
<td>14.5525</td>
<td>0.6217</td>
<td>1.1760</td>
<td>33.1091</td>
<td>20.7411</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>14.5525</td>
<td>0.4274</td>
<td>0.9816</td>
<td>54.0172</td>
<td>33.8389</td>
<td></td>
<td></td>
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<tr>
<td>$\gamma = 0.5$</td>
<td>14.5525</td>
<td>0.2823</td>
<td>0.8365</td>
<td>69.6314</td>
<td>43.6204</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the Student-$t$ copula model with $\nu = 5$, it is not possible to reach the true VaR = 14.5525% without lowering $\pi$ even for zero correlation.

the tails of the factors become, fair premiums are decreasing.

However, for both models and parameter settings under consideration, we do not observe price differences exceeding about ±10% for the same hedge range and plausible parameter constellations. Overall, we can conclude that the fair fees are quite robust on these alternative model specifications for the given hedge interval.

due to the inherent limitations of the Student-$t$ model dynamics being incapable to meet the target VaR without simultaneously lowering the probability of default. Thus, with a shifted expected loss in comparison to the other parameter settings, we are essentially observing a fundamentally different model.

Given that we are primarily concerned with the risk management of ‘real world’ credit losses during our analysis, we consider $\nu \approx 25$ and $\gamma \approx 5$ being plausible parameter constellations. The much lower values of $\nu$ and $\gamma$, typical for calibrated values from market expectations, would not fit to the historical default rates any more as they rather tend to the Gaussian limit.
3.4 Historical defaults, data quality, and estimation methods

We now apply our framework to historical one year default rates from the Moody’s annual default study 2015 (Ou et al., 2015). The use of one year default rates is supported by, e.g., the requirements of the current Basel Accords stating that “PD estimates must be a long-run average of one-year default rates for borrowers in the grade” (BCBS, 2006). We consider a 30 year time span ranging from 1985 to 2014. The number of rated companies documents an increase from 1,608 in 1985 to 5,340 in 2014. On average, we observe 3,590 rated companies per year. Table 4 provides descriptive statistics of the default rates for each rating grade.

Table 4: Summary statistics of one year corporate default rates. This table reports descriptive statistics for historical one year default rates in percent from the Moody’s annual default study 2015 (Ou et al., 2015). The sample period is from 1985 to 2014.

<table>
<thead>
<tr>
<th></th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa–C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>–</td>
<td>0.043</td>
<td>0.048</td>
<td>0.191</td>
<td>1.097</td>
<td>4.754</td>
<td>18.802</td>
</tr>
<tr>
<td>SD</td>
<td>–</td>
<td>0.166</td>
<td>0.114</td>
<td>0.319</td>
<td>1.204</td>
<td>4.179</td>
<td>13.249</td>
</tr>
<tr>
<td>Min.</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Med.</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.033</td>
<td>0.732</td>
<td>4.268</td>
<td>15.201</td>
</tr>
<tr>
<td>Max.</td>
<td>–</td>
<td>0.687</td>
<td>0.518</td>
<td>1.147</td>
<td>4.932</td>
<td>16.104</td>
<td>61.905</td>
</tr>
<tr>
<td>ø</td>
<td>113</td>
<td>418</td>
<td>836</td>
<td>786</td>
<td>485</td>
<td>724</td>
<td>228</td>
</tr>
<tr>
<td>#</td>
<td>–</td>
<td>6</td>
<td>14</td>
<td>46</td>
<td>151</td>
<td>762</td>
<td>1,035</td>
</tr>
</tbody>
</table>

ø (#) denote the average (total) number of rated (defaulted) debt

The mean of the historical one year default rates increases with rating grade, while the ratio of the mean and the standard deviation decreases. This observation supports the previous finding that high rated debt may be more prone to parameter risk than lower rated debt.

In contrast to the asymptotic ideal of the ASRF in the simulation study above, real world portfolios are not perfectly fine grained. Therefore we consider a Gaussian finite homogeneous pool model. Additionally, in each rating grade, there is at least one year with zero defaults, which makes the asymptotic case inapplicable. Another data driven consequence is that we employ two different estimation methods, since the low number of default data for high rated risk buckets is too sparse for conventional statistical inferences.
Guidance how to distinguish between low default and ‘normal’ default debt pools comes from, e.g., the FCA (2016) in BIPRU 4.3.95 “[...] a firm’s internal experience of exposures of a type covered by a model or other rating system is 20 defaults or fewer [...]”. For this reason, we apply the most prudent estimation principle detailed in Pluto and Tasche (2005) for the rating grades Aaa to A. For the remaining rating grades Baa to Caa–C, we apply the maximum likelihood estimation method outlined in Frey and McNeil (2003).

For the low default rating grades we infer the probability of default as an upper confidence bound for a given confidence level $\gamma$ such that the maximum value of $\hat{\pi}$ from the set of all admissible values of $\pi$ fulfills the inequality

$$1 - \gamma \leq \mathbb{P}[\text{No more then } k \text{ defaults observed}].$$

Here, we infer the upper bound of a confidence interval, $\pi_\gamma$, with the extension of the most prudent estimation principle to correlated default events in a multi-period setting. One restriction is that this estimation principle solely calibrates the probability of default. Thus we have to set $\rho$. In BCBS (2006) $\rho$ is floored and capped by 12% and 24%, respectively. That is why we determine the most prudent probabilities of default for these two values of $\rho$. To define the hedge range we apply the estimation principle to two different confidence levels of $\gamma$. As a proxy to an $\hat{\pi}$ estimate, we consider a value of $\gamma = 50\%$ and $\pi_h$ is approximated by $\gamma = 90\%$ to add some conservatism to our estimate and serves as the hedge parameter. Table 5 summarizes the estimated $\pi$ as functions of the confidence level $\gamma$. The third and fourth column show the estimates for $\pi$ corresponding to two correlation assumptions for each rating grade for two different levels of $\gamma$. These two upper confidence bounds proxy the parameter estimate and hedge parameter. They also span the hedge range shown in column six and seven for both the VaR and CVaR at confidence level $\alpha = 99.9\%$. The last two columns report the contractual fees $f^*_c$ for a parameter risk hedge in basis points.

Table 6 summarizes the results of the maximum likelihood approach (Frey and McNeil, 2003) for the rating grades Baa to Caa–C. The second and third column show
Table 5: Contractual fees for historical one year default rates for low default risk buckets. This table reports most prudent estimates for low default risk buckets Aaa to A. Upper confidence bounds for probability of default estimates for two confidence levels $\gamma \in \{50\%, 90\%\}$ are denoted by $\overline{\pi}_\gamma$. Contractual fees $f_c^*$ are for contract types $c \in \{0, 1\}$. The sample period is from 1985 to 2014.

<table>
<thead>
<tr>
<th>Rating</th>
<th>$\rho$ [%]</th>
<th>$\overline{\pi}_{50%}$ [%]</th>
<th>$\overline{\pi}_{90%}$ [%]</th>
<th>$R_{\alpha=99.9%}$</th>
<th>$R(\overline{\pi}_{50%})$ [%]</th>
<th>$R(\overline{\pi}_{90%})$ [%]</th>
<th>$f^*_c$ [bp]</th>
<th>$f^*_1$ [bp]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>12</td>
<td>0.0566</td>
<td>0.0818</td>
<td>VaR</td>
<td>1.7699</td>
<td>2.6549</td>
<td>0.1175</td>
<td>0.1512</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.0560</td>
<td>0.1119</td>
<td>VaR</td>
<td>3.5398</td>
<td>4.4248</td>
<td>0.0699</td>
<td>0.1553</td>
</tr>
<tr>
<td>Aa</td>
<td>12</td>
<td>0.0566</td>
<td>0.0879</td>
<td>VaR</td>
<td>1.4354</td>
<td>1.6746</td>
<td>0.0167</td>
<td>0.0394</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.0611</td>
<td>0.1165</td>
<td>VaR</td>
<td>2.6316</td>
<td>4.3062</td>
<td>0.1681</td>
<td>0.3149</td>
</tr>
<tr>
<td>A</td>
<td>12</td>
<td>0.0600</td>
<td>0.0987</td>
<td>VaR</td>
<td>1.1962</td>
<td>1.7943</td>
<td>0.0678</td>
<td>0.1140</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>0.0646</td>
<td>0.1306</td>
<td>VaR</td>
<td>2.6316</td>
<td>4.4258</td>
<td>0.1948</td>
<td>0.3696</td>
</tr>
</tbody>
</table>

the estimates for $\rho$ and $\pi$ for each rating class and their corresponding standard errors $\delta(\overline{\rho})$ and $\delta(\overline{\pi})$ in parenthesis. We hedge the estimates by a weighted standard deviation with $\kappa = 0.75$. Assuming that the estimator follows a normal distribution, the true parameter would then be less than or equal to this hedge level with a probability of 77.34%. The fifth and sixth column show the VaR and CVaR for a confidence level at 99.9% at the estimates $\theta$ and the hedge level $\theta_h$. Finally, the seventh and eights column report the contractual fees $f^*_c$ for a parameter hedge in basis points.

In Table 5 and Table 6 the estimates $\overline{\pi}$ are increasing with decreasing rating grades and compare well to the average default rates reported in Table 4. Due to the most prudent estimation principle, the probabilities of default for the first three ratings grades do not differ substantially and are increasing for the upper correlation level. For the rating grades Baa to Caa–C the asset correlation is also estimated and, with the exception of rating grade B, we observe that the asset correlations $\overline{\rho}$ are higher for higher rated risk buckets. For these rating classes the two risk measures at the estimates $R(\theta)$ are strictly increasing with declining credit quality.\textsuperscript{15} However, we do not see

\textsuperscript{15}The overall magnitudes of the risk measures may seem somewhat high, however, as noted above we do not consider nonzero recovery rates. On average, annual recovery is roughly about 38–42%, however,
Table 6: Contractual fees for historical one year default rates. This table reports maximum likelihood estimates $\hat{\theta} = [\hat{\rho}, \hat{\pi}]$ of each risk bucket for the rating grades Baa to Caa–C. The standard errors (in parentheses) for the parameter estimates are obtained by inverting the negative Hessian of the log-likelihood at the estimates. Risk measures VaR and CVaR at the estimate $\hat{\theta}$ and hedge level $\theta_h = \hat{\theta} + 0.75\hat{\sigma}(\hat{\theta})$ are evaluated for a confidence level of $\alpha = 99.9\%$. Contractual fees $f_c^*$ are for contract types $c \in \{0, 1\}$. The sample period is from 1985 to 2014.

<table>
<thead>
<tr>
<th>Rating</th>
<th>$\hat{\rho}$ [%]</th>
<th>$\hat{\pi}$ [%]</th>
<th>$\mathcal{R}_{\alpha=99.9%}(\hat{\theta})$ [%]</th>
<th>$\mathcal{R}(\theta_h)$ [%]</th>
<th>$f_0^*$ [bp]</th>
<th>$f_1^*$ [bp]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baa</td>
<td>14.7791***</td>
<td>0.1952***</td>
<td>VaR 3.4351</td>
<td>5.4707</td>
<td>0.2448</td>
<td>0.4432</td>
</tr>
<tr>
<td></td>
<td>(6.3334)</td>
<td>(0.0657)</td>
<td>CVar 4.5558</td>
<td>7.3357</td>
<td>0.1434</td>
<td>0.2407</td>
</tr>
<tr>
<td>Ba</td>
<td>13.9788***</td>
<td>1.1101***</td>
<td>VaR 11.5464</td>
<td>15.4639</td>
<td>0.3932</td>
<td>0.7657</td>
</tr>
<tr>
<td></td>
<td>(4.4327)</td>
<td>(0.2522)</td>
<td>CVar 13.9618</td>
<td>18.7691</td>
<td>0.2238</td>
<td>0.3928</td>
</tr>
<tr>
<td>B</td>
<td>23.4588***</td>
<td>4.9322***</td>
<td>VaR 43.2320</td>
<td>51.9337</td>
<td>0.7803</td>
<td>1.6501</td>
</tr>
<tr>
<td></td>
<td>(5.1883)</td>
<td>(1.0612)</td>
<td>CVar 49.2976</td>
<td>58.5884</td>
<td>0.3973</td>
<td>0.7555</td>
</tr>
<tr>
<td>Caa–C</td>
<td>13.8124***</td>
<td>17.9317***</td>
<td>VaR 60.9649</td>
<td>67.9825</td>
<td>0.8052</td>
<td>1.4774</td>
</tr>
<tr>
<td></td>
<td>(3.8477)</td>
<td>(2.0877)</td>
<td>CVar 65.0747</td>
<td>72.2711</td>
<td>0.3858</td>
<td>0.6766</td>
</tr>
</tbody>
</table>

*, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively

such behavior for the low default rating grades. The reasoning behind this is, that both risk measures in the correlated binomial model takes into account the absolute number of entities in each rating grade under consideration. Comparing Table 4, the highest rating grade has 113 entities, Aa rating grade has 418 entities and A has 836. With a greater number of entities idiosyncratic risk increasingly diversifies and consequently the risk measures decrease. However, if all three rating grades had the same number of entities, we would observe an increasing effect in the risk measures for these rating grades.

Given the sparsity of default events in the higher rated risk buckets, estimated standard errors are large in relation to the parameter estimates. By comparison, the (relative) standard errors for $\hat{\rho}$ are more pronounced than for $\hat{\pi}$. Note, however, the parameter sensitivity in $\pi$ of the hedge premiums is about twice than for $\rho$.\(^{16}\) Thus, there is no clearcut distinction between the importance of the parameters under consideration.

For the resulting risk measures, the values for the hedge parameters are higher than the values for the estimates by construction. If a financial firm were to interpret it is well documented that recovery rates tend to be quite volatile over time and are subject to the kind of provided bond collateral (secured, unsecured, subordinated), see, e.g., Altman et al. (2005).

\(^{16}\)Compare Section A.2 in the internet appendix for an illustration.
the hedge level as conservative parameter estimates and fully provide capital reserves
to reduce the probability of (unexpected) losses, this would come at high costs, e.g.,
compare for the Baa rated risk bucket a VaR_{\alpha}(\theta_h) = 5.4707\% to a VaR_{\alpha}(\hat{\theta}) = 3.4351\%.
If, instead, the same firm engages in a parameter hedge, it could achieve the same security level for a periodic contractual premium payment amounting to f_0^* = 0.2448 bp
or f_1^* = 0.4432 bp.\textsuperscript{17} Further, we see that the f_\alpha^* are consistently higher for VaR than for
CVaR, which is in line with the findings in Section 3.1.\textsuperscript{18} The calculated contractual hedge fees—based on the hedge level determined by the parameter estimation errors—tend to result in higher absolute premiums with lower rating grades. We find, however,
higher relative premiums \( f_\alpha^*/R(\theta_h) \) for higher rating grades. The relative premiums (in percent, \( c = 0 \)) for the Baa through Caa–C rating grades for the VaR are 0.0447,
0.0254, 0.0150 and 0.0118, respectively. Thus, we conclude, higher rated risk buckets are more prone to parameter risk than lower rated risks buckets. The same observation holds also for CVaR and the two contractual payoff types under consideration. This observation is in contrast to the findings from Section 3.1, where we assume the same (relative) error leading to higher relative fees for higher rating grades are smaller.

Summarizing, we find strong empirical support that in practical applications high
rated financial instruments are more prone to parameter risk than lower rated in-
struments, and—as a result—are more costly in relative terms to be hedged against
parameter risk. Further, we exemplify different estimation methods and highlight the
significant effects a departure from the ASRF ideal might have on real world credit
portfolios.

\textsuperscript{17}Though we regard A a low default risk bucket, we also estimate the corresponding maximum likelihood estimates (standard errors) yielding \( \hat{\rho} = 20.0577\% (13.2795\%) \), \( \hat{\sigma} = 0.0567\% (0.0310\%) \). The estimated correlation is well in between the 12\% and 24\% bounds we consider for the low default estimations. The hedge range is from VaR_{\alpha}(\hat{\theta}) = 1.9139\% to VaR_{\alpha}(\theta_h) = 4.1866\%. The upper hedge level is now more than double than the risk measure at the estimate. The hedge premiums would yield to \( f_0^* = 0.3153 \text{ bp} \) and \( f_1^* = 0.5276 \text{ bp} \).

\textsuperscript{18}For Aaa rated debt, however, we observe a peculiarity. The differences from \( f_\alpha^* \) for CVaR are rather high since both, the ratio of the hedge levels and the absolute range are quite pronounced. The fee for \( f_0^* \) is extremely low, whereas for \( f_1^* \) it seems more reasonable. This is traced back to the fact that \( f_0^* \) merely insures the range from 2.6511\% to 3.1240\% which can only happen if three entities default.
4 Conclusion

The possibility that inferred model parameters deviate from the true model parameters—termed parameter risk—may significantly affect calculated risk measures. We introduce a framework allowing economic agents to hedge this parameter risk. It is easily applicable to arbitrary types of risk, is independent from the model type, chosen hedge interval, and works for any risk measure.

We show that the possibility to hedge potential parameter errors up to a certain level may drastically reduce costs in comparison to provisions. This may have important economic implications in light that additional capital requirements may influence institutional’s behavior and potentially provide adverse incentives. Further, we demonstrate that prospective protection sellers may get into the unique position to diversify parameter risk. This may have significant policy implications as the consequences from beneficial diversification might provide positive macro side effects to an economy as a whole.

Future applications could include other loss categories than credit risk like, e.g., market risk or operational risk. Additionally, it would be of great relevance to compare different estimation methods and survey feasible approaches to identify appropriate hedge levels. The framework may further be a promising addition to the toolset for model validation, discriminate between parameter and model risk or analyze effects of hedged parameters on stress testing.

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A Internet appendix to: Hedging parameter risk

A.1 Parameter risk in the asymptotic single risk factor model

Underpinning the Basel internal ratings-based (IRB) approach (BCBS, 2006), the ASRF model is well known to academics and practitioners. Its foundation and derivation is given by Vašíček (1987, 1991, 2002) and for rigorous treatments of this model compare Gordy (2000, 2003).

To gain intuition about parameter risk in the ASRF model, we perform the following simulation study. For the asset correlation, we assume $\rho = 20\%$, which lies in between the lower and upper bounds found in the Basel Accords. The probabilities of default are $\pi \in \{0.2\%, 1\%, 5\%\}$, roughly corresponding to historical one year default rates of Moody’s Baa, Ba, and B rated corporates (compare Table 4). To analyze effects of differing sample sizes, we consider three time horizons ranging from a short period of $T = 7$ years, over $T = 15$, up to a time period of $T = 30$ years. For each case, we sample credit losses according to Equation (4). During each iteration, we employ the estimation method described in Duellmann et al. (2010) providing analytical solutions for the estimates $\hat{\theta} = [\hat{\rho}, \hat{\pi}]$ with corresponding standard errors $\hat{\sigma}(\hat{\theta})$. Based on these results, we define the hedge parameter as the sum of the estimate and its standard error weighted by a confidence factor $\kappa = 75\%$ (arguments on the choice and definition of such a hedge parameter follow below). For each estimate $\hat{\theta}$ and hedge parameter $\theta_h$, we calculate the value-at-risk (VaR) at the 99.9% confidence level. This procedure is repeated $10^5$ times for each parameter set and time horizon. Figure A.1 summarizes the results.

The upper and middle panel illustrate how parameter risk affects estimates of the asset correlation $\hat{\rho}$ and the probability of default $\hat{\pi}$ for three different assumptions of $\pi \in \{0.2\%, 1\%, 5\%\}$ and time horizons $T \in \{7, 15, 30\}$, respectively. Each subplot depicts from the left to right the estimates ($\hat{\theta}/\theta$, dark gray), standard errors ($\hat{\sigma}(\hat{\theta})/\theta$, gray) and hedge parameters ($\theta_h/\theta$, light gray) normalized by the true parameter $\theta \in \{\rho, \pi\}$. The lower panel displays the normalized VaR for the estimates (VaR($\hat{\theta}$)/VaR($\theta$), dark gray).
Figure A.1: Impact of parameter risk on probability of default, correlation, and value-at-risk. This figure presents how parameter risk affects estimates of the correlation $\rho$ (first row), probability of default $\pi$ (second row), and each with the corresponding value-at-risk measure (third row) at confidence level $\alpha = 99.9\%$. For all three columns, the true correlation is given by $\rho = 20\%$. The true probabilities of default are increasing from the left column with $\pi = 0.2\%$, $\pi = 1\%$ in the middle column, and $\pi = 5\%$ in the right column. Each subplot individually reports the normalized estimation results for three time periods given by $T \in \{7, 15, 30\}$ years. Hedge parameters are defined according to Equation (A.1) by $\theta_h = \hat{\theta} + \kappa \hat{\sigma}(\hat{\theta})$, where $\hat{\theta}$ denote the estimate and $\hat{\sigma}(\hat{\theta})$ denote the corresponding standard error given the estimation approach in Duellmann et al. (2010). The confidence factor is set to $\kappa = 0.75$. The simulation is repeated $10^5$ times. The boxes are confined by the lower and upper quartiles and contain horizontal lines which draw the median values. Vertical extensions depict 1.5 times the interquartile range from the median, values outside these fences (whiskers) are considered outliers and plotted individually with a dot. Additionally, the circles within the box plots depict the mean values.
and the hedge parameters \(\text{VaR}(\theta_h)/\text{VaR}(\theta)\), light gray).

In each repetition, the sampled loss is a random event. Therefore the estimates \(\hat{\theta}\), standard errors \(\delta(\hat{\theta})\), hedge parameters \(\theta_h\) and value-at-risk are stochastic, and their distributions describe the parameter risk. These parameter risk distributions are displayed in the form of Tukey box plots where the circles depict the mean value in Figure A.1. For instance, for the shortest period of 7 years and the lowest probability of default of 0.2\% the median (mean) of the asset correlations’ parameter risk distribution is roughly 82\% (87\%) of the true value. The lower and upper quartiles range between 56\% and 112\%. More precisely, in 66\% of all repetitions, the estimate \(\hat{\rho}\) results in an underestimation of the true asset correlation \(\rho\).

With decreasing time span \(T\), the range of parameter estimates, their standard errors for both the correlation and probability of default as well as resulting VaR increase. The true parameters are—in the median—underestimated. On average, however, the inferred probabilities of default overestimate the true probability of default, which is caused by few, but severe, outliers. Whereas the correlation estimates are below the true parameter, both on average and in the median. In these cases, the estimated VaR figures underestimate the value-at-risk, both on average and in the median. The described effects are particularly pronounced for seemingly safe risk buckets linked to low probabilities of default. Summarizing, the ASRF model displays considerable parameter risk, at least on average, for short time horizons and low probabilities of default.

Next, we illustrate the effects of parameter risk for individual scenario realizations. This is particularly relevant from the perspective of a single financial institution, since they have to deal with one sole VaR estimation. Such an estimate could be seen as the result of a single realization of the VaR distributions for the estimates depicted in the lower panel of Figure A.1 by the dark gray box plots. Therefore we examine whether a simulated loss for subsequent periods under the true default generating process is greater than the anticipated VaR or not. These number of exceedances (NoE) are then accumulated. If the true VaR were anticipated at all iterations, the NoE must equal \(1 - \alpha\)
by definition of the VaR. We compare the relative NoE (%NoE) with the NoE permitted by the true model. Hence, if the ratio is greater than one, the VaR is insufficient to cover the losses encoded by $\alpha$. Table A.1 reports the %NoE for confidence levels $\alpha = 99\%$ and $\alpha = 99.9\%$ for the estimates and hedge parameters.

**Table A.1: Number of exceedances for value-at-risk at estimates and hedge parameters.** This table reports the relative number of exceedances (%NoE) for given confidence levels, i.e., $\frac{\text{NoE [%]}}{1-\alpha}$, with $\alpha \in \{99\%, 99.9\%\}$. The true correlation is $\rho = 20\%$. The upper panel reports the %NoE given the NoE are based on the value-at-risk at the estimates $\text{VaR}(\hat{\theta})$. The lower panel reports the %NoE given the NoE are based on the value-at-risk at the estimates plus one standard error times the confidence factor $\kappa = 0.75$, i.e., $\text{VaR}(\theta_h)$ is determined via Equation (A.1). The simulation is repeated $10^6$ times.

<table>
<thead>
<tr>
<th>$\pi$ [%]</th>
<th>T = 7</th>
<th>T = 15</th>
<th>T = 30</th>
<th>T = 7</th>
<th>T = 15</th>
<th>T = 30</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimates</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>4.5298</td>
<td>2.3582</td>
<td>1.6118</td>
<td>18.3455</td>
<td>5.9541</td>
<td>2.8357</td>
</tr>
<tr>
<td>1.0</td>
<td>4.5246</td>
<td>2.3475</td>
<td>1.6008</td>
<td>18.3932</td>
<td>5.9074</td>
<td>2.8150</td>
</tr>
<tr>
<td>5.0</td>
<td>4.5208</td>
<td>2.3536</td>
<td>1.6144</td>
<td>18.3488</td>
<td>5.8908</td>
<td>2.8131</td>
</tr>
<tr>
<td><strong>Hedge, $\kappa = 0.75$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>2.3785</td>
<td>1.2106</td>
<td>0.8991</td>
<td>8.3860</td>
<td>2.4541</td>
<td>1.2302</td>
</tr>
<tr>
<td>1.0</td>
<td>2.2872</td>
<td>1.1594</td>
<td>0.8643</td>
<td>8.1334</td>
<td>2.3124</td>
<td>1.1748</td>
</tr>
<tr>
<td>5.0</td>
<td>2.1807</td>
<td>1.1043</td>
<td>0.8357</td>
<td>7.7846</td>
<td>2.2053</td>
<td>1.1223</td>
</tr>
</tbody>
</table>

We find that the probability of default has a limited impact on the %NoE. This is in contrast to the time dimension. Reported %NoE in the upper panel based on the estimates show a substantial increase in the %NoE for decreasing $T$. This increase in %NoE gets worse the further in the tail one is trying to predict, e.g., the %NoE for $T = 7$ is roughly 2.5 (6.5) times higher than the %NoE at $T = 30$ at the $\alpha = 99\%$ ($\alpha = 99.9\%$) level, respectively. For example, the 18-fold tearing of the true allowed %NoE for $T = 7$ interprets as follows. Given the chosen estimation method and an estimation horizon of seven years, the resulting estimated VaR lead to 18 times more breachings of the intended VaR level than it would be the case if instead using the true parameters. This %NoE implies a VaR of an effective confidence level of 98.16% instead of 99.9%. That is, an estimation approach, which would imply an estimation approach leading in $\frac{1}{3}$ of the cases to the true VaR, in $\frac{1}{3}$ of the cases to an underestimation by 25% and in $\frac{1}{3}$ of the cases to an overestimation, results in the same %NoE. This essentially signifies
that an equiprobable overestimation is by no means able to even out an equiprobable underestimation particularly in the rare cases.

This analysis is further used to motivate an appropriate hedge level. Given the risk measure under consideration is monotone increasing in the parameters (for this chosen model type), the hedge parameter is defined as

$$\theta_h = \hat{\theta} + \kappa \hat{\sigma}(\hat{\theta}),$$  \hspace{1cm} (A.1)

where $\kappa$ is a confidence factor describing the influence of estimation error $\hat{\sigma}(\hat{\theta})$ on the hedge level. We find $\kappa = 0.75$ to be a good candidate to encode the influence of estimation errors on hedge parameters to reach a fair level of conservatism. That is, with the resulting hedge level, the %NoE are roughly cut by half and for $T = 30$ is almost approaching the $1 - \alpha$ value. Though our choice may seem rather ad hoc, more elaborate methods are conceivable (compare, e.g., Tarashev (2010)). For instance, the confidence factor could be inferred from the distribution of (robust) standard errors, or could be calibrated in a QIS like impact study from regulatory authorities. The FCA (2016) in BIPRU 4.3.88 explicitly demands that “A firm must add to its estimates a margin of conservatism that is related to the expected range of estimation errors. Where methods and data are less satisfactory and the expected range of errors is larger, the margin of conservatism must be larger”. Similarly, the confidence factor could also be seen as reflecting regulators’ confidence in the employed model, compare, e.g., BCBS (2013). For our analysis, we rely on Equation (A.1) for its analytical tractability and its clearcut interpretation.

### A.2 Parameter sensitivity of fair hedge premiums and relation of fair hedge premiums to contractual fees

The isolines in Figure A.2 connect pairs of parameter estimates sharing the same fair hedge premium $f_c$ for the two risk measures VaR and CVaR at confidence level $\alpha = 99.9\%$, as well as the two contract types $c \in \{0, 1\}$. A $-45^\circ$ isoline would represent
Figure A.2: Parameter sensitivity of fair hedge premiums. This figure plots the fair hedge premiums $f_c, c \in \{0, 1\}$ in basis points, for the risk measures VaR and CVaR, given the true parameter constellation $\rho = 20\%, \pi = 1\%$, and confidence level $\alpha = 99.9\%$. The estimate $\hat{\rho}$ vary along the abscissa, while the estimate $\hat{\pi}$ vary along the ordinate.
a situation where the two estimates $[\hat{\rho}, \hat{\pi}]$ have equal influence on the fair premiums, however, we can see that the sensitivity in $\pi$ is about twice as high than it is for $\rho$. This pattern is stable for all four cases, otherwise the contours share pretty similar and stable shapes, only the levels change, i.e., $f_1$ are strictly higher than for $f_0$ for both risk measures as expected.

To get an impression how the fair hedge premiums with known true values $f_c$ relate to contractual fees $f_c^*$ we look at following setup. For example for $\alpha = 99.9\%$, we assume $\rho = 20\%$ and $\pi = 1\%$. This results to a $\text{VaR}^G_\alpha(\rho, \pi) = 14.55\%$. Further assume an estimate $[\hat{\rho}, \hat{\pi}] = [15\%, 0.75\%]$ implying a $\text{VaR}^G_\alpha(\hat{\rho}, \hat{\pi}) = 9.01\%$.

![Figure A.3: Relation of fair hedge premiums to contractual fees.](image)

The results of this setup are depicted in Figure A.3. Clearly, if $\hat{\theta} = \theta_h$, then $f_c = f_c^* = 0$, because there is no hedge contract. Further, in cases with a positive effective
error and $\theta_h = \theta$, we see $f_c = f^*_c > 0$. Even if $\theta_h \neq \theta$, it is possible that $f_c = f^*_c > 0$. But in this case, of course, the fee is not the true fee, because the hedge level changed. For $\hat{\theta} \leq \theta_h \leq \theta$ the difference is $f_c - f^*_c \geq 0$, while for $\theta_h \geq \theta$ the difference is $f_c - f^*_c \leq 0$. Further, we note a positive maximum for the difference $f_c - f^*_c$ around $\rho < \rho_h < \rho$ for small $\pi_h$.

The protection buyer has a strong incentive to engage in a hedge, if $f_c - f^*_c \geq 0$, which holds particularly for the quadrant $\hat{\theta} \leq \theta_h \leq \theta$. The incentive for selling protection is diametral, i.e., is highest for $f_c - f^*_c \leq 0$ in the area described by $\theta_h \geq \theta$.

### A.3 Alternative model specifications

For model inference, we use the same maximum likelihood approach and data detailed in Section 3.4. In a first step, we fit the model parameter $\rho$ and $\pi$ simultaneously with the hyperparameter $\nu$ and $\alpha$. For the Student-$t$ model the estimated degrees of freedom $\nu$ is about 25, while for the NIG shape parameter $\alpha$, we find a value about 5. This results clearly indicate that the asset return distributions inferred from historical default rates are not as heavy tailed as typically result from calibration to market data. Further, we note that the estimates of the hyperparameter and the correlation parameter $\rho$ are highly correlated. A possible explanation could be that they both heavily influence the skewness of the loss distributions. For this reason, we fix the parameters to $\nu = 15$ and $\alpha = 3$. Thus, we force the models to exhibit slightly more heavier tails than supported by empirical data to better exemplify the impact of heavy tailed models. Then, subject to these fixed hyperparameter, we repeat the estimation of $\rho$ and $\pi$. Finally, based on the estimation errors of the parameters we then define the hedge level $\theta_h = \hat{\theta} + 0.75 \hat{\sigma}(\hat{\theta})$, obtain the corresponding $\text{VaR}_{\alpha=99.9\%}$ and calculate the contractual fees. The results of this estimation procedure are reported in Table A.2.

The Gaussian, Student-$t$ and NIG models show considerable variation in the resulting VaR for each risk bucket. With decreasing rating grade the relative differences between the VaR are decreasing, too. Overall, the Student-$t$ and NIG models typically lead to higher VaR which seems reasonable, since with heavier tails, *ceteris paribus*...
Table A.2: Contractual fees for historical one-year default rates for alternative model specifications. This table reports $\hat{\theta} = [\hat{\rho}, \hat{\pi}]$ of each risk bucket for the rating grades Baa to Caa–C for three different ASRF model specifications. The first row for each rating category shows the results for the Gaussian copula model as the limiting case for both the Student-$t$ and NIG models. The second row shows the results for the Student-$t$ model with fixed degrees of freedom $\nu = \nu_B = 15$. The third row shows the results for the NIG model with fixed shape parameter $\alpha = 3$. Based on the parameter estimates, the VaR at the estimates and the VaR at the hedge level $\theta_h = \hat{\theta} + 0.75 \delta(\hat{\theta})$ are reported for the confidence level $\alpha = 99.9\%$. The last two columns show the contractual fees $f^*_c$ in basis points. The sample period is from 1985 to 2014.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Model</th>
<th>$\hat{\rho}$ [%]</th>
<th>$\hat{\pi}$ [%]</th>
<th>VaR($\hat{\theta}$) [%]</th>
<th>VaR($\theta_h$) [%]</th>
<th>$f^*_c$ [bp]</th>
<th>$f^*_t$ [bp]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baa</td>
<td>$\nu = \alpha = \infty$</td>
<td>14.7792**</td>
<td>0.1952***</td>
<td>3.4351</td>
<td>5.4707</td>
<td>0.2448</td>
<td>0.4432</td>
</tr>
<tr>
<td></td>
<td>$\nu = 15$</td>
<td>0.0001</td>
<td>0.2588***</td>
<td>5.9796</td>
<td>7.3791</td>
<td>0.0698</td>
<td>0.2016</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 3$</td>
<td>16.3660***</td>
<td>0.2015**</td>
<td>4.5802</td>
<td>7.8880</td>
<td>0.3286</td>
<td>0.6575</td>
</tr>
<tr>
<td>Ba</td>
<td>$\nu = \alpha = \infty$</td>
<td>13.9788***</td>
<td>1.1101***</td>
<td>11.5464</td>
<td>15.4639</td>
<td>0.3932</td>
<td>0.7657</td>
</tr>
<tr>
<td></td>
<td>$\nu = 15$</td>
<td>0.0100</td>
<td>1.1795***</td>
<td>11.7526</td>
<td>17.3196</td>
<td>0.8886</td>
<td>1.4139</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 3$</td>
<td>15.1828***</td>
<td>1.1403***</td>
<td>14.4330</td>
<td>20.2062</td>
<td>0.5055</td>
<td>1.0715</td>
</tr>
<tr>
<td>B</td>
<td>$\nu = \alpha = \infty$</td>
<td>23.4588***</td>
<td>4.9322***</td>
<td>43.2320</td>
<td>51.9337</td>
<td>0.7803</td>
<td>1.6501</td>
</tr>
<tr>
<td></td>
<td>$\nu = 15$</td>
<td>15.0279**</td>
<td>4.8368***</td>
<td>40.6077</td>
<td>49.4475</td>
<td>0.8637</td>
<td>1.7323</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 3$</td>
<td>25.0968***</td>
<td>5.1423***</td>
<td>52.3481</td>
<td>63.1215</td>
<td>0.8435</td>
<td>1.9123</td>
</tr>
<tr>
<td>Caa–C</td>
<td>$\nu = \alpha = \infty$</td>
<td>13.8124***</td>
<td>17.9317***</td>
<td>60.9649</td>
<td>67.9825</td>
<td>0.8052</td>
<td>1.4774</td>
</tr>
<tr>
<td></td>
<td>$\nu = 15$</td>
<td>11.3962***</td>
<td>17.9661***</td>
<td>60.5263</td>
<td>67.5439</td>
<td>0.8323</td>
<td>1.5203</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 3$</td>
<td>14.2876***</td>
<td>17.7821***</td>
<td>65.3509</td>
<td>73.2456</td>
<td>0.7836</td>
<td>1.5424</td>
</tr>
</tbody>
</table>

*, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

the VaR should increase. For the speculative rating grades B and Caa–C, we see an exception for the Student-$t$ model.\textsuperscript{19} In general, the parameters appear to be plausibly estimated. Correlation estimates $\hat{\rho}$ around zero for the first two risk buckets Baa and Ba, may seem somewhat unusual, are, however, well supported by empirical evidence, see, e.g., Hamerle and Rösch (2005). Further, we find that, in general, the VaR($\theta_h$) from the Gaussian model cover all VaR($\hat{\theta}$) of the other models with more sophisticated factor distributions. Thus, a parameter risk hedge with a Gaussian hedge implicitly insures against model risk with respect to point estimates from the alternative model specifications. Analyzing the contractual premiums of a parameter risk hedge for real default data for a number of loss models show that the prices are quite robust, share similar properties and consistent behavior. On the other hand, different models typically coincide with different hedge levels or parameter ranges.

\textsuperscript{19}This exception mirrors the issues with the Student-$t$ model reported in Table 3 and briefly discussed in footnote 13.
References


FCA (2016), Prudential sourcebook for Banks, Building Societies and Investment Firms, Release 4, Financial Conduct Authority, United Kingdom.


