An Accurate Lattice Model for Pricing Catastrophe Equity Put under the Jump-Diffusion Process

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Abstract

A catastrophe equity put (CatEPut) is constructed to recapitalize an insurance company that suffers huge compensation payouts due to catastrophic events (CEs). The company can exercise its CatEPut to sell its stock to the counterparty at a predetermined price when its accumulated loss due to CEs exceeds a predetermined threshold and its own stock price falls below the strike price. Much literature considers the evaluations of a CatEPut that can only be exercised at maturity; however, most CatEPuts can be exercised early so the company can receive timely funding. This paper adopts lattice approaches to evaluate CatEPuts with early exercise features. To solve the combinatorial exposition problem due to the trigger of CatEPuts’ accumulated loss, our method reduces the possible number of accumulated losses by taking advantage of the closeness of integral additions. We also identify and alleviate a new type of nonlinearity error that yields unstable numerical pricing results by adjusting the lattice structure. We provide a rigorous mathematical proof to show how the proposed lattice can be constructed under a mild condition. Comprehensive numerical experiments are also given to demonstrate the robustness and efficiency of our lattice.

Keywords: catastrophic events, lattice, option, pricing

1 Introduction

An insurance company receives premiums from buyers and grants them the right to receive compensation against loss of property, health, and so on. To hedge the potential obligations resulting from insurance claims, insurance companies can diversify payment risks by enlarging the pool of policyholders or hedge risks by buying financial protections from reinsurers. However, the huge compensation payments caused by frequent catastrophic events (CEs) such as hurricanes, earthquakes, or even

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the 911 plane hijackings may exceed insurer ability to diversify payment risks and exceed the capacity of reinsurance markets. These huge payments could cause an insurance company to fail to meet a prespecified risk-based capital ratio required by financial regulators or even endanger the company’s continuing operations. To stabilize the financial status of insurance companies, various derivatives such as catastrophe equity puts (CatEPuts) and bonds have been developed to absorb the huge losses caused by CEs. The design and evaluation of such derivatives are thus critical for both academic studies and insurance industries.

To obtain timely fund injections for use in compensating CE losses, an insurance company can pay premiums to market investors for purchasing CatEPuts. The insurance company has the right to exercise the CatEPuts by selling its own stock at a predetermined strike price to the market investor, once its accumulated loss due to CEs exceeds a predetermined threshold $L$ and its stock price is lower than the strike price. CatEPut evaluation was pioneered by [4] in which the company’s stock price process is modeled using a lognormal diffusion process plus constant jumps to reflect loss caused by CEs. Many studies have introduced complex mathematical models to capture changes in CE occurrence intensities and loss magnitudes (e.g., [2, 13, 22]). However, most such work considers only so-called “European-style” CatEPuts that can only be exercised at the maturity date. The timely financial injection advantage resulting from exercising “American-style” CatEPuts early has not been well studied in the literature. Lo et al. [16] address this problem with Monte Carlo simulations but the method is inefficient and the resulting pricing results are probabilistic. It is also difficult for Monte Carlo simulations to determine the value under which CatEPuts are to be unexercised and hence the best strategy for exercising CatEPuts or not. On the other hand, Lin and Wang [15] and Kim et al. [12] study perpetual CatEPuts; however, to our knowledge, in practice, CatEPuts all have finite maturities.

An American-style derivative can be efficiently evaluated using lattice methods. A lattice divides the time span over the life of the derivative contract into $n$ time steps and simulates the evolution of the price process of the derivative’s underlying asset. For example, we can build a lattice to simulate an insurance company’s stock (or the underlying asset) price process and use the lattice to evaluate a CatEPut (or the derivative) owned by the company. Numerical pricing results generated by lattice methods converge to theoretical values with the increment of $n$ [8]. However, evaluating a complex derivative, like CatEPuts, poses a combinatorial exposition problem and the unsmooth convergence problem. The major contribution of this paper is to evaluate CatEPuts with our novel lattice that simultaneously addresses the aforementioned two problems detailed as follows.

The combinatorial exposition problem is due to the accumulated loss trigger covenant; that is, a CatEPut cannot be exercised until the accumulated losses contributed by CEs exceed the threshold $L$. Pricing derivatives with accumulations of values governed by a random process is analogous to the evaluation of Asian-style options, which is a long standing, hard computational problem. This is because the possible partial sums of values governed by a random process and hence the computation time may grow explosively with the number of time steps $n$ [5]. To address this problem, in this paper, each CE loss is modeled by an integral multiple $k$ of a basic jump unit $h$ (discussed later); thus the
accumulated loss can again be interpreted as an integral multiple of $h$ due to closeness of the integral additions and the number of possible accumulated losses drop dramatically. To calibrate the statistical properties of CE losses, the magnitude of both $k$ and $h$ are tuned to match the first few moments of the loss distribution as suggested by [10].

Instead of converging smoothly, pricing results generated using lattice methods can converge erratically or even oscillate significantly due to the so-called nonlinearity error [9], i.e., the pricing error due to the nonlinearity of a derivative’s value function. This problem is commonly alleviated by adjusting the lattice structure such that certain lattice node(s) match the underlying asset’s price levels where the derivative’s value function is highly nonlinear, as suggested in [1, 6, 19, 20, 21]. However, for the evaluation of CatEPuts, we discover that the accumulated-loss trigger of CatEPuts constitutes a new type of nonlinearity error that cannot be alleviated by simply adjusting the lattice structure; specifically, the discretization of a CE loss also leads to the nonlinearity error once none of integral multiples of $h$ matches the threshold $\mathcal{L}$, where the CatEPut’s value function is also highly nonlinear. To address this newly identified problem, we present a method to adjust the size of the basic jump unit $h$ such that one of its integral multiples coincides with $\mathcal{L}$. Note that the statistical properties of CE losses must be simultaneously calibrated to ensure that the pricing results generated by our lattice still converge to the theoretical value of the CatEPut. The mathematical proof is derived to show how the proposed lattice can be constructed to satisfy the aforementioned two requirements under a mild condition. Comprehensive numerical experiments are also given to demonstrate the robustness and efficiency of our lattice.

Our paper is organized as follows: The mathematical models and background financial knowledge for CatEPus are introduced in Section 2. In Section 3, we first review how to construct lattices for both the lognormal diffusion process and the jump-diffusion process; then we introduce the nonlinearity error when pricing with lattice methods in Section 3. In Section 4 we propose a sophisticated lattice for pricing CatEPuts that simultaneously solves the combinatorial exposition problem and the nonlinearity error problem. Numerical results are given in Section 5 that show the robustness and efficiency of our lattice. Section 6 concludes the paper.

2 Modeling and Definitions

In this section we show that the stock price process with jumps due to CEs proposed in [11] can be reduced to a jump-diffusion process, which can be efficiently and robustly modeled by the lattices proposed in [10, 7]; thus we adapt their lattices to price CatEPuts with smooth convergence and efficiency. These two price processes and the definitions of CatEPuts are detailed in this section as follows.
2.1 The Mathematical Model of the Jump-Diffusion Process

To model the leptokurtic features like high peaks and heavy tails of a stock return’s distribution, the stock price at time $t$ $S_t$ is usually modeled by a jump-diffusion process denoted as \[ S_t = S_0 e^{(r - \lambda \bar{k} - 0.5 \sigma_s^2) t + \sigma_s W_t + J(t)} \] (1), where the jumps $J(t) = \prod_{i=0}^{n(t)} (1 + k_i)$ and $k_0 = 0$. In the above equation, $W_t$ denotes a standard Brownian motion, $r$ denotes the risk-free rate, and $\sigma_s$ denotes the volatility of the diffusion component of the price process. Jump events are governed by the Poisson process $n(t)$ with jump intensity $\lambda$, where $n(t)$ denotes the number of jumps that occur at or before time $t$. The lognormal jump magnitude $k_i (i > 0)$ satisfies the following distribution: $\ln (1 + k_i) \sim N(\gamma J, \delta^2)$, where $\bar{k} = E(k_i) = e^{\gamma + 0.5 \delta^2} - 1$. The diffusion component, the random jump magnitude, and the Poisson process are assumed to be independent [18].

Let $S_0$-log price of $S_t$ (denoted as $V_t$) be the logarithm of $S_t$ dividing by $S_0$. Both Hilliard and Schwartz [10] and Dai et al. [7] decompose $V_t$ into the diffusion component and the jump component by rewriting Equation (1) as $V_t \equiv \ln \left( \frac{S_t}{S_0} \right) \equiv X_t + Y_t$, where the diffusion component is $X_t \equiv (r - \lambda \bar{k} - 0.5 \sigma_s^2) t + \sigma_s W_t$, and the jump component is $Y_t \equiv \sum_{i=0}^{n(t)} \ln (1 + k_i)$.

2.2 The Stock Price Process with Jumps due to CEs

This paper considers the constant interest rate version of the stock price process proposed by [11] for pricing CatEPuts as $S_t = S_0 e^{(r-0.5\sigma_s^2) t + \sigma_s W_t - \alpha(L(t) - kt)}$, where the accumulated loss $L(t) = \sum_{i=1}^{n(t)} \ell_i$ denotes the compound Poisson process and $\{\ell_i : i = 1, 2, \ldots\}$ are iid random variables denoting the size of the $i$-th loss with the probability density function $f_L(y)$. The term $kt$ compensates for the presence of $L(t)$ to keep stock return being equal to the risk-free rate $r$ under the risk-neutral measure. It is derived as [11] $E\left[ e^{-\alpha(L(t) - kt)} \right] = 1 \Rightarrow k = \frac{\lambda}{\alpha} \int_{-\infty}^{\infty} (1 - e^{-\alpha y}) f_L(y) dy$. Recall that $\lambda$ is the jump intensity of the Poisson process $n(t)$. 

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2.3 Catastrophe Equity Put Options (CatEPuts)

A European-style CatEPut can only be exercised at the maturity date $T$ with payoff $\mathcal{P}(T)$ as [15]

$$\mathcal{P}(T) = \mathbb{1}_{\{L(T) - L(0) > \mathcal{L}\}} \max ((X - S_T), 0),$$  \hspace{1cm} (5)

where $L(T) - L(0)$ denotes the accumulated losses of the insurance company over the period $[0, T]$, $\mathcal{L}$ is the loss threshold specified in the contract, and $X$ is the strike price. An American-style CatEPut can be exercised prior to time $T$; its exercised payoff function at time $t$ $\mathcal{P}(t)$ is defined as

$$\mathcal{P}(t) = \begin{cases} X - S_t, & L(t) - L(0) > \mathcal{L} \text{ and } X - S_t > CV_t, \\ 0, & \text{otherwise,} \end{cases}$$  \hspace{1cm} (6)

where $CV_t$ denotes the continuation value of the CatEPut at time $t$ if it is left unexercised. Continuation values can be calculated by backward induction in lattice methods. Obviously, the insurance company exercises the CatEPut if it would be more beneficial than keeping it, under the condition that the accumulated losses have exceeded the contract-specified threshold $\mathcal{L}$.

3 Preliminaries

In this section we describe the lattice structures for modeling the lognormal diffusion and the jump-diffusion processes that are used in this paper. Then we discuss the nonlinearity error that causes the lattice to produce unstable results.

3.1 The Lattice Structures for the Lognormal Diffusion Process

The evolution of the lognormal diffusion part in Equation (2) is usually modeled by the CRR binomial lattice [3], as illustrated in Figure 1(a). Specifically, a CRR lattice divides a certain time span, say $[0, T]$, into $n$ discrete time steps and specifies the movements of the stock price between two adjacent time steps. The length of each time step $\Delta t$ is equal to $T/n$ and the stock price $S_t$ at time $t$ moves upward to $S_t u$ with probability $P_u$ and downward to $S_t d$ with probability $P_d = 1 - P_u$ at time $t + \Delta t$, where

$$P_u = \frac{e^{\sigma \sqrt{\Delta t}} - d}{u - d},$$

$u = e^{\sigma \sqrt{\Delta t}}, d = 1/u$. The aforementioned branching probabilities and multiplication factors are set to match the first two moments of Equation (2). Note that the ratio between any two adjacent nodes can be calculated as $S_t u / S_t d = e^{2\sigma \sqrt{\Delta t}}$.

The trinomial structure illustrated in Figure 1(b) can be inserted into the CRR structure to adjust the lattice structure or to deal with the invalid probability problem [17]. In this paper we take advantage of this structure to connect the jump nodes caused by the jump component back to the CRR lattice to reduce the number of lattice nodes. Let the nodes at time $t + \Delta t$ follow the CRR lattice layout (i.e., the
Figure 1: **Lattice structures for the lognormal diffusion process.** The length of a time step is $\Delta t$ for both panels. Panel (a) illustrates a three-time-step CRR Lattice. In panel (b), $\mu$ and $\hat{\mu}$ denote the expected stock return and the return to reach node B, respectively. In addition, $\alpha$, $\beta$, and $\gamma$ denote the logarithmic returns to reach nodes A, B, and C from $S_t$ minus the expected return $\mu$, respectively (e.g., $|\alpha| = \ln(S_{t+\Delta t}^A/S_t) - \mu) = \hat{\mu} + 2\sigma_s \sqrt{\Delta t} - \mu)$, where $S_{t+\Delta t}^A$ denotes the stock price at node A at time $t+\Delta t$. The branching probability for each branch is listed next to the branch and the log-price difference between two adjacent nodes at the same time step is $2\sigma_s \sqrt{\Delta t}$.

difference of logarithmic stock prices between any two adjacent nodes stays at $2\sigma_s \sqrt{\Delta t}$.) Define the one-time-step expected stock return $E(\ln(S_{t+\Delta t}/S_t))$ as $\mu$. The middle branch (with probability $p_m$) emitted from $S_t$ connects to the node, say B in this case, whose one-step return $\hat{\mu}$ is closest to $\mu$ among all the nodes at time $t+\Delta t$—hence this is called the “mean-tracking” method. The upper branch (with probability $p_u$) and the lower branch (with probability $p_d$) connect to the adjacent nodes of B, say A and C in this diagram. Define $\alpha$, $\beta$, and $\gamma$ as the stock price returns from $S_t$ to nodes A, B, and C minus the expected stock return $\mu$, respectively. Dai and Lyuu argue that valid branching probabilities can be derived by solving the following equations [6]:

\begin{align}
 p_u \alpha + p_m \beta + p_d \gamma &= 0, \quad (7) \\
p_u(\alpha)^2 + p_m(\beta)^2 + p_d(\gamma)^2 &= \text{Var}, \quad (8) \\
p_u + p_m + p_d &= 1, \quad (9)
\end{align}

where the first and second moments of the stock return are matched in Equations (7) and (8), respectively. Equation (9) ensures that the sum of all branching probabilities equals one.

### 3.2 The Lattice for the Jump-Diffusion Process

To model the jump-diffusion process, Dai et al. propose a state-of-the-art $O(n^{2.5})$ lattice, the structure of which can be adjusted to coincide with where the derivative’s value function is highly nonlinear
to alleviate the nonlinearity error such that the pricing results converge smoothly [7]. In their lattice, the diffusion component in Equation (2) is modeled by a mixture of the aforementioned CRR and trinomial lattice structures, where the nodes on such a mixture lattice are denoted as diffusion nodes. On the other hand, the jump component is modeled by positioning \((2m + 1)\) jump nodes to match the first \(2m\) local moments of the distribution of lognormal stock price jumps occurring within a time step as in [10]. Then the one-step stock returns of these \((2m + 1)\) nodes can be expressed as \(c \sigma_s \sqrt{\Delta t} + jh\), where \(c = \pm 1\) and \(j = 0, \pm 1, \pm 2, \ldots, \pm m\). Note that \(c = -1, 1\) denotes the upward or the downward movement of the stock price, respectively, driven by the diffusion component, and the accumulated jump size within a time step is discretized as the basic jump unit \(h\) times an integral multiple \(j\); \(j = 0\) denotes that no jumps occur within a time step. The magnitude of the basic jump unit is set to

\[ h = \sqrt{\gamma^2 + \delta^2}. \]

The probabilities \(q_j\) corresponding to the discrete jump size \(jh\) are chosen to match the first \(2m\) local moments of the sum of stock price jump within a step. Formally, we set

\[ \sum_{j=-m}^{j=m} (jh)^i q_j = \mu_i' \equiv E \left[ \sum_{w=0}^{n(\Delta t)} \ln(1 + k_w) \right]^i, \quad i = 1, 2, \ldots, 2m, \]

where \(\mu_i'\) is the \(i\)-th local moment of \(Y_{\Delta t}\) (see Equation (3)). In addition to the above \(2m\) equations, the sum of probabilities

\[ \sum_{j=-m}^{j=m} q_j = 1 \]

must hold. By solving the above \(2m + 1\) equations, we obtain \(2m + 1\) probabilities \(q_j\) \((j = 0, \pm 1, \pm 2, \ldots, \pm m)\). Since the diffusion component and the jump component are assumed to be independent, the probabilities for the one time step return becoming \(\sigma_s \sqrt{\Delta t} + jh\) and \(-\sigma_s \sqrt{\Delta t} + jh\) are \(P_u q_j\) and \(P_d q_j\), respectively.

Figure 2 illustrates how Dai et al. [7] combine the CRR binomial lattice and the trinomial lattice illustrated in Figure 1 to model the jump-diffusion process by adopting the aforementioned jump node positioning scheme suggested in [10]. Each time step is decomposed into two phases: the diffusion phase (for the diffusion component Equation (2)) and the jump phase (for the jump component Equation (3)). In the diffusion phase, a diffusion node at time step \(\ell - 1\) with stock price \(S\) moves upward to \(Su\) with probability \(P_u\) and downward to \(Sd\) with probability \(P_d\) (the CRR lattice structure); in the ensuing jump phase, the node with stock price \(Su\) can jump to \(Sue^{jh}\), and the node with stock price \(Sd\) can jump to \(Sde^{jh}\) with probability \(q_j\) mentioned above, where \(-m \leq j \leq m\). These nodes that are (or are not) affected by the jump events within time step \(\ell - 1\) are jump (or diffusion, respectively) nodes. Then, in the diffusion phase of the next time step \(\ell\), the outgoing branches from diffusion nodes (marked by white circles) still follow the CRR structure to connect to diffusion nodes, while the outgoing branches from jump nodes (marked by gray circles) connect back to diffusion nodes by the trinomial structures (marked by dashed lines) depicted in Figure 1. Our paper then modifies this lattice to price CatEPuts (introduced later in Section 4).
Pricing options is accomplished on the lattice by so-called backward induction. Define $F(S, i)$ as the option value given that the stock price is $S$ at time step $i$. The option value at time step $n$ (i.e., at maturity) equals the option payoff $\mathcal{P}(T)$. For European-style options, the value on the node with stock price $S$ at time step $i$ can be evaluated as the discounted expected option values at the following time step as

$$F(S, i) = e^{-r\Delta t} \sum_{j=-m}^{m} \left[ F \left( S e^{\sigma \sqrt{\Delta t} + jh, i+1} \right) P_u q_j + F \left( S e^{-\sigma \sqrt{\Delta t} + jh, i+1} \right) P_d q_j \right].$$

For pricing American-style options, holders determine whether to keep the option alive (with the value calculated by backward induction) or to exercise it immediately (as in Equation (6)) to maximize their benefits. The backward induction for pricing CatEPuts is more complex due to the trigger of accumulated losses as in Equations (5) and (6).

### 3.3 Nonlinearity Errors

Although the price generated by the lattice converges to the theoretical option value as $n \to \infty$ [8], the price may converge unstably due to nonlinearity error [9]. This error is due to the the nonlinearity of the option value function. For example, the value of a CatEPut can be viewed as the function of the underlying stock price, the current time (step), and the accumulated losses. This function is nonlinear when the accumulated losses are around $L$ and when the stock price is around the strike price $X$ (see Equations (5) and (6)). Nonlinearity errors can usually be much reduced by making a node or a price level of the lattice match where the option value function is highly nonlinear [6]. Although we can adjust the lattice to have a price level to coincide with the strike price $X$ to alleviate parts of the nonlinearity error by the method proposed in [6], the error caused by the nonlinearity due to accumulated losses cannot be reduced by such adjustment. In this paper we identify this new type of the nonlinearity error and solve the problem in the following section.

### 4 Lattice Construction

#### 4.1 Coefficient Comparisons between Stock Price Processes

To price CatEPuts by taking advantage of the jump-diffusion-process lattice illustrated in Figure 2, we rewrite the jump-diffusion process in Equation (1) and the catastrophe model in Equation (4) as

$$S_t = S_0 e^{(r-0.5\sigma^2_s)t + \sigma_s W_t - \lambda \bar{k} t + \sum_{i=1}^{n(t)} \ln(1+k_i)}$$

and

$$S_t = S_0 e^{(r-0.5\sigma^2_j)t + \sigma_j W_t + \alpha t - \sum_{i=1}^{n(t)} \alpha t_i},$$

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Figure 2: The jump-diffusion lattice and the allocation of loss states. The white and gray circles represent the diffusion and jump nodes, respectively. Each time step $\Delta t$ is divided into two phases: diffusion and jump. The dashed lines represent the trinomial structure that is used to connect a jump node to three diffusion nodes during the diffusion phase, and $m$ is set to one in this figure for simplicity. The numbers in red beside each node denote the possible loss states for that node. The blue curves denote two different paths for the accumulated losses. The jump parameters $\gamma_J$ and $\delta_J$ are set to $-0.4$ and $0.16$, respectively; thus the jump magnitude $h$ is $0.430813$. 

$$h = \sqrt{\gamma_J^2 + \delta^2} = \sqrt{(-0.4)^2 + 0.16^2} = 0.430813$$
respectively. Here we assume that the loss $\ell_i$ satisfies the normal distribution $\ell_i \sim N(\gamma_\ell, \delta_\ell)$. Then we compare the coefficients of above two equations and express $\gamma$, $\delta$, and $\bar{k}$ in the jump-diffusion process with the coefficients in the catastrophe model as
\[
\begin{align*}
\gamma_J &= -\alpha \gamma_\ell, \\
\delta^2 &= \alpha^2 \delta^2_\ell, \\
\bar{k} &= -\frac{\alpha k}{\lambda}.
\end{align*}
\]

### 4.2 Jump-Diffusion Lattice Construction and Loss State Allocation

The aforementioned coefficient comparisons allow us to reduce a catastrophe model to a jump-diffusion one and construct a jump-diffusion lattice with $O(n^{2.5})$ nodes by modifying the lattice proposed by [7] as shown in Figure 2 for pricing CatEPuts. However, the accumulated CE losses embedded in a CatEPut’s payoff function (see Equations (5) and (6)) influence the CatEPut value; therefore, extra loss states are required in each lattice node to record different CatEPut values under different accumulated losses.\(^1\) For example, node F can be reached via paths A-D-F and B-D-F; each path entails a different accumulated loss and hence a different CatEPut value at node F. To prevent the number of possible accumulated losses from growing explosively, each CE loss is modeled by an integral multiple $k$ of a basic jump unit $h$; thus the accumulated loss can again be interpreted as an integral multiple of $h$ due to closeness of the integral additions. Figure 2 gives an illustrative example of the possible states (numbers in red) in a 2-time-step sample lattice, in which the numbers of possible jump sizes in the jump phase are set to $2m + 1$ with $m = 1$ (i.e., 3) for simplicity. As shown in the figure, each node at time step 1 has only one state, and there are 3 types of loss states: 1, 0, and $-1$. At time step 2, there are more possible states for each node since each node can be reached by different price paths with different historical price jumps. For example, node F can be reached by paths A-D-F (state +2) and B-D-F (state +1). Under this setting, the minimum and the maximum jump magnitudes for a node at time step $i$ are $-(i - 1)m \times h$ and $(i - 1)m \times h$, respectively. The closeness of integral additions entails that there are $2(i - 1)m + 1$ possible jump magnitudes for that node. Thus our pricing algorithm runs in $O(n^{3.5})$ as discussed later.

### 4.3 Inaccurate Prices Due to Nonlinearity Errors

Recall that the nonlinearity error is due to the error introduced by the nonlinearity of the CatEPut value function, such as the strike price $X$ and the accumulated loss trigger $\mathcal{L}$ in Equation (5). The nonlinearity error caused by the accumulated-loss trigger cannot be alleviated by simply adjusting the lattice structure as suggested in [6, 7, 9]; specifically, the discretization of a CE loss also leads to the nonlinearity error once none of integral multiples of $h$ matches the threshold $\mathcal{L}$, where the CatEPut

\(^1\)In a later section, the jump branches are adjusted to calibrate the loss distribution; thus all jump magnitudes shall be all negative to reflect the stock price decrements due to CE losses.
value function is highly nonlinear. For example, the accumulated losses of the two blue paths in Figure 2 are 0.430813 and 0.86162, respectively; in this case, changing the thresholds $L$ from 0.431 to 0.86 does not change the payoff of these two price paths and hence the CatEPut’s evaluation result. As discussed in Section 5, this unreasonable phenomenon turns the CatEPut’s pricing results into a stepwise function of $L$.

4.4 Adjustments for the Jump-Diffusion Lattice

4.4.1 Jump Magnitude Adjustment

The setting of the basic jump unit $h = \sqrt{\gamma_j^2 + \delta^2}$ fails to match $L$ as illustrated in Figure 3 and thus causes a new type of non-linearity error mentioned in Section 4.3. To address this newly identified problem, we present a method to adjust the size of the basic jump unit from $h$ to $h'$ to make one of its integral multiples coincide with $L$. The modified basic jump unit is set to

$$h' = \frac{L}{a},$$

(12)

where $a = \lceil L/(\omega \delta) \rceil$, $\delta$ is the standard deviation of $\ln(1+k_i)$ (see Section 2.1), and $\omega$ is a positive integer selected as 2 as in Theorem 4.1. A constructive proof given in Section 4.4.3 shows that the aforementioned settings help us to construct a valid trinomial branch under a mild condition to calibrate the distribution of CE losses so that the pricing results generated by our lattice still converge to a theoretical CatEPut’s value.

4.4.2 Jump Branch Adjustments

In traditional lattice methods for modeling the jump-diffusion process [10, 7], the jump branches are symmetric; that is, the number of branches below the middle branch (i.e., no jump) is the same as that above the middle branch (see Equation (11) and Figure 2). However, as stock price losses due to CEs are usually relatively large, the branch structure should be adjusted to calibrate the loss distribution per Figure 4(b). Our modified lattice considers the three-branch case (i.e., $m = 1$): the adjusted middle branch for modeling the jump component in the jump-diffusion process is selected to be an integral multiple of $h'$ (denoted as $-\kappa h'$) that is closest to the jump mean $\gamma_j$ (see the “mean-tracking” concept.
Figure 4: **Jump branch adjustments.** Panel (a) depicts the jump branches for the original jump-diffusion lattice; panel (b) shows the modified jump branches for calibrating stock price jumps due to CEs. The stock price $S_t$ reaches nodes A, B, and C with probabilities $p_u$, $p_m$, $p_d$, respectively.

in Figure 1(b)); we have

$$\kappa = \left\lfloor -\frac{\gamma J h'}{h'} + 0.5 \right\rfloor.$$

Figure 4 compares the traditional jump branches with the modified ones, in which the diffusion nodes (denoted by white circles) discretely model the evolution of the lognormal diffusion process (i.e., stock process without jump events), while the jump nodes (denoted by gray circles) model the stock process with jump events alone.

### 4.4.3 Branching Probability Adjustment

To ensure that the pricing results generated by our lattice converge to the theoretical CatEPut value, our lattice modified from [7] must calibrate the stock price with CE losses per Equation (4). Although the branch modifications in Sections 4.4.1 and 4.4.2 resolve the nonlinearity error problem and calibrate the loss’s distribution, respectively, these modifications may lead to the invalid branching probability problem (see [17]). To address this problem, we here adopt the trinomial structure introduced by [6] (see Figure 1(b)) to calibrate the branching probabilities for the modified jump branches.

As shown in Figure 4(b), the adjusted middle branch emitted from the diffusion node with price $S_t$
to the jump node B is selected as the closest integral multiple of $h'$ to the jump mean $\gamma_J$. The logarithmic returns to reach nodes B, A, and C from $S_t$ minus the jump mean $\gamma_J$ are defined as

$$ \beta = -\kappa h' - \gamma_J, $$

$$ \alpha = \beta + h', $$

$$ \gamma = \beta - h', $$

respectively, where $\beta \in [-h'/2, h'/2]$ and $\alpha > \beta > \gamma$. The three branching probabilities can be obtained by solving Equations (7)–(9), where Var in Equation (8) is replaced with $\delta^2$, the variance of the jump magnitude. These three branching probabilities are solved by Cramer’s rule as

$$ p_u = \det(u)/\det, \quad (13) $$

$$ p_m = \det(m)/\det, \quad (14) $$

$$ p_d = \det(d)/\det, \quad (15) $$

where

$$ \det = (\beta - \alpha)(\gamma - \beta)(\gamma - \alpha), \quad (16) $$

$$ \det(u) = (\beta \gamma + \delta^2)(\gamma - \beta), \quad (17) $$

$$ \det(m) = (\alpha \gamma + \delta^2)(\alpha - \gamma), \quad (18) $$

$$ \det(d) = (\alpha \beta + \delta^2)(\beta - \alpha). \quad (19) $$

The following theorem shows that our branch construction method yields valid branches under a mild condition.

**Theorem 4.1** The proposed modification with $\omega = 2$ in Equation (12) results in valid branching probabilities in Equations (13)–(15) under the mild condition $\mathcal{L} > 4\delta$.

**Proof:** Note that all valid branch probabilities should lie in the interval $[0, 1]$. Under the premise $p_u + p_m + p_d = 1$ in Equation (9), it suffices to show that $p_u, p_m, p_d \geq 0$. Since $\alpha > \beta > \gamma$, we have $\det < 0$; thus all branching probabilities derived in Equations (13)–(15) are non-negative under the premise that $\det(u), \det(m), \det(d)$ derived in Equations (17)–(19) are non-positive. Since $\alpha > \beta > \gamma$, we have

$$ \beta \gamma + \delta^2 \geq 0, \quad \alpha \gamma + \delta^2 \leq 0, \quad \alpha \beta + \delta^2 \geq 0. \quad (20) $$

For derivation purposes, we define $\hat{h} = \omega \mathcal{L}$ and rewrite the denominator of Equation (12) as

$$ a = \left[ \frac{\mathcal{L}}{\hat{h}} \right]; $$

therefore, we have

$$ (a - 1)\hat{h} < \mathcal{L} \leq a\hat{h} $$

$$ \iff (a - 1)\hat{h} < ah' \leq a\hat{h} $$

$$ \iff 1 \leq \frac{\hat{h}}{h'} < \frac{a}{a - 1} \text{ for } a > 1. \quad (21) $$

13
We first prove \((\beta \gamma + \delta^2) \geq 0\) by replacing \(\gamma\) and \(\delta\) with \((\beta - h')\) and \(\hat{h}/\omega\), respectively, as follows:

\[
\beta \gamma + \delta^2 = \beta(\beta - h') + \left(\frac{\hat{h}}{\omega}\right)^2 \geq 0
\]

\[
\Leftrightarrow \left(\frac{\hat{h}}{\omega}\right)^2 \geq \beta(h' - \beta), \quad (22)
\]

where the maximum value of the right-hand side of Equation (22) \((h'/2)^2\) occurs when \(\beta = h'/2\). Substituting this maximum value for the right-hand side of inequality (22) yields

\[
\left(\frac{\hat{h}}{\omega}\right)^2 \geq \frac{h'}{2}, \quad (23)
\]

\[
\Leftrightarrow \frac{\hat{h}}{\omega} \geq \frac{h'}{2}, \quad (24)
\]

\[
\Leftrightarrow \omega \leq \frac{2\hat{h}}{h'}. \quad (25)
\]

Above, inequality (23) is a sufficient condition of inequality (22), and inequality (24) holds since \(h', \hat{h}, \omega \geq 0\). Due to \(\hat{h}/h' \geq 1\) derived in inequality (21), we have \(\omega \leq 2\) as a sufficient condition of inequality (25), and thus inequality (22): \((\beta \gamma + \delta^2) \geq 0\). Similarly, the sufficient condition for the inequality \((\alpha \beta + \delta^2) \geq 0\) can be also derived as \(\omega \leq 2\). For the inequality \((\alpha \gamma + \delta^2) \leq 0\), we have

\[
(\alpha \gamma + \delta^2) = (\beta + h')(\beta - h') + \left(\frac{\hat{h}}{\omega}\right)^2 \leq 0
\]

\[
\Leftrightarrow \left(\frac{\hat{h}}{\omega}\right)^2 \leq h'^2 - \beta^2, \quad (26)
\]

where the minimum value of the right-hand side of Equation (27) is \(3h'^2/4\) as

\[
-\frac{1}{2}h' \leq \beta \leq \frac{1}{2}h' \Leftrightarrow 0 \leq \beta^2 \leq \frac{1}{4}h' \Leftrightarrow \frac{3}{4}h'^2 \leq h'^2 - \beta^2 \leq h'^2.
\]

Substituting the minimum value for the right-hand side of inequality (27) yields

\[
\left(\frac{\hat{h}}{\omega}\right)^2 \leq \frac{3h'^2}{4}
\]

\[
\Leftrightarrow \omega \geq \frac{2\hat{h}}{\sqrt{3}h'}. \quad (28)
\]

Again, inequality (28) holds since \(h', \hat{h}, \omega \geq 0\). Since \(\hat{h}/h' < a/(a - 1)\) as in (21), we have \(\omega \geq 2a/(\sqrt{3}(a - 1)) \approx 1.1547a/(a - 1)\) as a sufficient condition of inequality (28), and hence inequality (26): \((\alpha \gamma + \delta^2) \leq 0\).
Since the aforementioned derivations show that the sufficient conditions for yielding valid probabilities are \( \omega \leq 2 \) and \( \omega \geq 2a/(\sqrt{3}(a-1)) \), we select \( \omega = 2 \) to maximize the feasible range of \( a \). Substituting \( \omega = 2 \) into the second sufficient condition yields \( a \geq \sqrt{3}/2 \approx 2.366 \), which is equivalent to the condition \( \mathcal{L} > 4\delta \) due to the relation \( a = \left\lceil \frac{\mathcal{L}}{\omega \delta} \right\rceil \) mentioned in Equation (12). This condition is mild since a typical CatEPut contract cannot be triggered until a series of CEs have occurred to significantly influence an insurance company’s financial status.

5 Numerical Results

In this section we assess the accuracy, efficiency, and robustness of the proposed lattice for pricing CatEPuts. First, we compare the prices of CatEPuts generated by the traditional and proposed lattices, and illustrate how our method alleviates nonlinearity errors. Then we compare the values of European-style and American-style CatEPuts to analyze the value differences due to early exercise features. We then provide sensitivity analyses on several jump-related or contract-related parameters to examine the robustness of our lattice. Finally, we examine the time complexity and the convergence property.

The numerical settings of the base case are as follows: the initial stock price \( S_0 = 40 \), the strike price \( X = 30 \), the stock return volatility \( \sigma_s = \sqrt{0.05} \), the interest rate \( r = 0.08 \), the time to maturity \( t = 0.25 \) (years), the jump parameters \( \lambda = 5 \), \( \gamma_J = -0.6 \), \( \delta = \sqrt{0.05} \), the loss threshold \( \mathcal{L} = 2 \), and the number of time steps \( n = 200 \). In addition, the benchmark European-style CatEPut values are computed using Monte Carlo simulation with 10,000 simulated paths.

Figure 5(a) first compares the prices generated by the traditional and proposed lattices with various loss thresholds ranging from 1.00 to 2.80. The purple and the blue curves denote the prices of the European-style CatEPuts generated by the traditional lattice and by our modified lattice, respectively, whereas the gray dotted curves denote the 95% confidence intervals of the simulated results. Observe that while the prices generated by the traditional lattice converge non-smoothly like a step function (also mentioned in Section 4.3), the proposed lattice produces smooth and accurate pricing results that are consistent with the benchmark results generated using Monte Carlo simulation. For the red curve, that in panel (a) denotes the prices of American-style CatEPuts, which are difficult to evaluate with simulation methods. Panel (b) lists the early exercise premium, i.e., the price difference between the American- and European-style CatEPuts. Since this premium can occupy more than half of a CatEPut’s value, ignoring the early exercise feature (as practiced in much literature) significantly misprices CatEPuts.

Sensitivity analyses for changing the strike price \( X \), the jump magnitude parameters \( \gamma_J \), \( \delta \), and the jump intensity \( \lambda \) on both European- and American-style CatEPuts are presented in Figure 6. The blue and red curves denote the prices of European- and American-style CatEPuts generated by the proposed lattice method, respectively. The price differences between the two reflect the early exercise premium, which can be very large under the scenarios discussed below. The dash curves denote the upper and
(a) Prices generated by traditional and proposed lattices

![Graph showing the comparison of traditional and proposed lattices for different loss thresholds.](image)

(b) Early exercise premiums of American-style CatEPuts

<table>
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<tr>
<th>$L$</th>
<th>European-style Price</th>
<th>American-style Price</th>
<th>Early exercise premium Value</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.2584</td>
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<td>1.2</td>
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<tr>
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<td>0.3398</td>
<td>0.5113</td>
<td>0.1715</td>
<td>33.54%</td>
</tr>
</tbody>
</table>

Figure 5: **Pricing CatEPuts with different loss thresholds.** In panel (a), the purple and blue curves denote the prices of the European-style CatEPuts generated by the traditional and proposed lattice, respectively; the gray dotted curves denote the 95% confidence interval of CatEPut values generated using Monte Carlo simulation; and the red curve denotes the prices of American-style CatEPuts. Panel (b) tabulates the values of European- and American-style CatEPuts in the second and third columns; the fourth and fifth columns list the early exercise premium and the proportion of the premium to the CatEPut value, respectively. Numerical settings follow the base case except for the loss threshold $L$ illustrated in the first column.
Figure 6: **Sensitivity Analyses for European and American CatEPuts.** Numerical settings follow the base case except for the setting specified by the x-axis of each figure.

Panel (a) suggests that increments in the strike price $X$ raise the CatEPut values; this is intuitive since increments in $X$ increase the CatEPut payoff as illustrated in Equations (5) and (6). In addition, the early exercise premium tends to grow with increments in the strike price $X$.

The statistical properties of CEs, including the mean and the standard derivation of the jump magnitude and the jump intensity, also significantly influence CatEPut values, as illustrated in Panels (b), (c), and (d), respectively, of Figure 6. This is because a CatEPut can be exercised once the threshold $L$ is exceeded by the accumulated loss, which tends to grow as the jump magnitude mean $\gamma_J$ becomes more negative or as the jump intensity $\lambda$ grows. That is why (both European- and American-style) CatEPut values and the early exercise premiums grow with decrements in $\gamma_J$ (or increments in $\lambda$). Next, the lower bounds of the 95% confidence intervals of European-style CatEPuts generated using Monte Carlo simulations. The robustness of our lattice method is confirmed by observing that all European-style CatEPut values generated by our lattice fall within the confidence intervals under various scenarios.
jump magnitude standard derivation $\delta$ measures the variations in CE losses; increments in $\delta$ entail a greater chance of enormous CE loss and consequently a higher likelihood of a larger accumulated losses as well as a larger CatEPut value.

Finally, we examine the time complexity and the convergence property of our lattice in Figure 7. Panel (a) plots the logarithm of the execution time versus the logarithm of the number of time steps. As an empirical estimate of the order of the time complexity, the 3.447 slope of the regression line is consistent with our theoretical analysis of $O(n^{3.5})$. Recall that our lattice is modified from the state-of-the-art $O(n^{2.5})$ jump-diffusion lattice proposed by [7] and that accumulated losses are modeled by inserting loss states which can be expressed by integral multiples of a basic jump unit. With this design we avoid the combinatorial explosion problem with accumulated features embedded in options, say Asian options. The number of loss states grows polynomially at $O(n)$ as stated in Section 4.2, which results in the proposed $O(n^{3.5})$ lattice. Panel (b) shows that the pricing results of our model converge smoothly to the benchmark value with the increment in the number of time steps $n$.

6 Conclusions

In this paper we propose a lattice method that evaluates CatEPuts with early exercise features (i.e., American-style CatEPuts) without incurring the combinatorial explosion or nonlinearity error problems. We provide a rigorous mathematical proof to show how the proposed lattice can be constructed under the mild condition $\mathcal{L} > 4\delta$. The numerical results confirm the accuracy and robustness of our
lattice for pricing European-style CatEPuts (i.e., without early exercise features). In most past work only European-style CatEPuts are considered; however, we show that ignoring early exercise features yields significant pricing errors.

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References


