The Share of Systematic Risk in Foreign Exchange and Stock Markets

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Abstract

Is there a common systematic risk between the foreign exchange market and the stock market? To answer this question, a two-country affine model of exchange rates is proposed to obtain a Forex factor. We show that this factor is an important driver of the stock market risk premium. Not only it contributes a sizable portion of exchange rate volatility, but also outperforms the commonly-used financial and macroeconomic variables in terms of predicting stock excess returns. The predictive power is robust with respect to forecasting horizons and different characteristic portfolios. In addition, the cross-sectional study shows that the Forex factor has substantial explanatory power for cross-section return differences of industry portfolios, the performance is better than Fama-French three factor model and is comparable with that of the up-to-date five factor model.

JEL classification: E43, F31, G12, G17

Keywords: Affine term structure model, Particle filter, Exchange rate, Stock return predictability

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1. Introduction

Uncovering common sources of systematic risk from different markets is of crucial importance for international asset pricing and policy analysis. A rich strand of the literature has documented the integration of the stock markets around the world (see a recent review by Lewis (2011)). If stock markets of different countries are not segmented, the equilibrium equity price is partially determined by a world common stochastic discount factor (SDF). Intuitively, this same logic should also line up the financial markets of different asset classes. However, unlike the relatively consistent results on the relation between different stock markets, the empirical evidence on the share of systematic risk between the foreign exchange and the stock market is mixed. For example, Jorion (1991) finds that the currency risk is almost negligible in the stock market, whereas Carrieri et al. (2006) reports the significant currency risk premium. In particular, a recent paper by Burnside (2012) finds that the successful factor models in the literature that have been shown to well explain one market have little explanatory power for the other market. Such empirical results are puzzling since if a risk factor based on one market is indeed informative about the stochastic discount factor of investors, then it should also have pricing implications for other markets.

Instead of directly examining the well-established risk factors in the foreign exchange or the stock market, in this paper we use a different strategy in that we search for the plausible factor through an affine term structure model of interest rate and exchange rate. Such a model has received much attention in modeling the bond yield (see e.g. Ang and Piazzesi (2003)) and there is growing interest in extending it to describe the exchange rate (see e.g. Backus et al. (2001) and Anderson et al. (2010)). The common objective of using such model is to evaluate the impact of the observable or latent states, which may be plausible state variables of the SDF, on the bond yields and exchange rates. However, here we shift our focus to study the usefulness of the underlying states of SDF in explaining the stock market.

The goal of this paper is twofold. First, we propose and estimate an affine model of the joint dynamics of exchange rate and interest rate. In addition to the commonly used risk factors for capturing the bond yield movements, we use a latent state to capture the fluctuations of the exchange rate and the implied variance from currency options. We evaluate the empirical performance of such model and discuss how bond risk factors and the latent state affect the exchange rate and the currency options. Second and more importantly, we examine whether this extracted latent state from the foreign exchange market is an important risk factor for the stock market. Not only we test whether the latent state is a significant driver of the time-varying expected return of the aggregate stock market, but also we discuss its relevance in explaining the cross-sectional return differences of the industry portfolios.
We find that the latent factor in the estimated SDF, which we term as the Forex factor, turns out to be a strong predictor of the home and foreign country aggregate stock market risk premia. The slope coefficient of the predictive regression for home (foreign) market is -0.97% (-0.67%), with a t-statistic 2.72 (2.28) and adjusted $R^2$ 4.16% (2.15%). Moreover, the predictability is statistically significant for most horizons ranging from 1-month to 36-month. For home (foreign) market, the adjusted $R^2$ changes from 3.83% (2.15%) for two-year (one-month) horizon to 9.51% (12.5%) for three-month (three-year) horizon. Such a factor is also important in driving the time-varying expected return of different characteristic portfolios, which have been shown by many papers to have distinct risk profiles (see e.g. Petkova (2006)). The predictive power is significant for most of the characteristic portfolios constructed in Fama and French (2015). Besides the predictability at the time-series dimension, the factor also contributes to explaining the cross-sectional differences in average returns of industry portfolios. Adding the Forex factor greatly enhances the pricing ability of CAPM or Fama-French three-factor model on the cross-sectional industry portfolios, which is a challenging task as shown in Lewellen et al. (2010). Even the original CAPM model augmented with the Forex factor now performs similarly to the recently proposed Fama and French (2015) five-factor model, with comparable adjusted $R^2$ 41.9% and 42.7% respectively. Notice that the above implications for the stock market is achieved when the Forex factor is required to reconcile the fluctuations of exchange rate. Therefore, the results here strongly support the close connection between the foreign exchange and the stock markets.

There are three main contributions in this paper. First and foremost, our work is similar to Atanasov and Nitschka (2015) in the sense that we both study the common source of systematic risk between those two markets. While they uncover the integration by exploring the effect of discount rate and cash-flow news of stock return on the carry trade portfolios, we use data of exchange rate to estimate the pricing kernels within a no-arbitrage affine model and investigate the implications of a key factor (Forex factor) in the estimated pricing kernel on the stock market. Therefore, our work can be treated as a complementary to theirs. Also, their study relies on the ICAPM framework. Thus, they assume that the SDF is a linear function of exogenous risk factors. Instead, we will show that the parametric form of the SDF model we use is an equilibrium outcome, and the risk factor is endogenously estimated from the exchange rate and the interest rate data. In addition to the implications for the aggregate stock market, this paper also studies the pricing of currency risk in different industry portfolios, similar to Francis et al. (2008). Again, their currency risk factor is exogenously constructed instead of endogenously estimated. Another difference is that we evaluate the impact of risk factor on cross-section industry portfolio returns in a simple

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1The data of Forex factor is available upon request
unconditional factor model, while they use a conditional model. Although the conditional model has the advantage of incorporating the time-varying investment opportunity set, its empirical implementation and results are possibly sensitive to the selection of conditioning variables.

Second, this paper contributes to the literature on the joint modeling of exchange rate and interest rate (see e.g. Inci and Lu (2004), Anderson et al. (2010)). While previous papers restrict their attention in those two markets, we show that the implication can be extended to other asset markets. Hence we bridge the gap between the literature of term structure and stock return predictability. Also, most previous studies use models whose states are all latent to explain the interest rate and exchange rate, one exception is Yin and Li (2014), who use models with all states being observed. Both approaches have shortcomings. On the one hand, the latent factors have trouble mapping directly to macroeconomic interpretations. On the other hand, since exchange rate is far more volatile than many macroeconomic quantities, the model with all macro states may result in bad fit. In this paper, we combine those two approaches by working with a model where most states are all observables, but we introduce one latent state to account for the volatile exchange rate. We show that the estimated model can replicate almost perfectly the movement of exchange rate return, yet retain the satisfactory yield curve fit by the classical affine term structure model. The model performance is remarkable since Sarno et al. (2012) indicates that for many models there is a substantial trade-off between the accuracy of yield curve fitting and exchange rate return.

Third, this article extends the discussions in Corradi et al. (2013) to the foreign exchange market by studying how exchange rate volatility and implied volatility changes in response to macroeconomic states in an internally consistent no-arbitrage model. The literature on the determinants of exchange rate volatility such as Devereux and Lane (2003) mainly uses the regression approach together with regressors motivated by the economic theory. Instead, we adopt an asset pricing approach, and relate the exchange rate to the pricing kernels in a no-arbitrage manner. In addition, we also discuss the driving forces of currency option implied volatility.

The rest of the paper is organized as follows. In Section 2, we discuss the model setting, and how it differs from existing approaches in the literature. Section 3 lays out the data used and discusses our econometric framework. Section 4 reports the model estimation results and the evaluation of fit. Section 5 bridges the foreign exchange market and stock market through the return predictability exercises and discusses the potential sources of predictability. Section 6 explores some additional implications and Section 7 concludes.
2. Model

The model is a two-country extension of the macro-factor term structure model in Ang and Piazzesi (2003) and Joslin et al. (2014), where the home and foreign country are U.S. and U.K. respectively. Such model has been shown by numerous literature to be capable of well capturing the bond yield. To take into account the exchange rate data, in addition to observable states for each country, we add one latent state (Forex factor) that only affects the exchange rate but not bond yield, neither in the spanned nor unspanned way. This way of modeling provides the flexibility of fitting the data since exchange rate return is far more volatile than bond yield.

2.1. State Dynamics

We include two kinds of observable states into the model. First, since it has been widely accepted that yield curves can be well characterized by a small number of factors, we use portfolios of bond yields, i.e. the first two principal components of the yield curve, as one class of observables. Those two states have clear interpretations as the level and slope factor, and they account for around 99% of cross sectional bond yield variations in the sample studied here. Adding higher order principal factors contributes little to the model fit, whereas the number of parameters will explode.

In addition, we include inflation and industrial production growth into the observable states. There are two reasons to consider those factors. First, mounting evidence documents the existence of unspanned risk in bond market (see e.g. Duffee (2011)). That is, the bond risk premia can’t be well explained by the cross-section of yields, but can be explained by variables that do not contribute to the cross-section fit of yield curve. Joslin et al. (2014) find that the measures of inflation and growth have large effects on bond risk premia, and thus shall be good candidate for unspanned risk factors. Second, the effects of those two states for exchange rate have been well studied in the literature (see e.g. Engel (2014) for a recent review). A number of theoretical models find that inflation and economic growth can be quite relevant for understanding the fluctuations of nominal exchange rate, thus it will be interesting to investigate the empirical performance of such states in tracking the exchange rate movements.

The observable states in two countries are highly correlated with each other, with correlations ranging from 0.5 to 0.99. To facilitate the interpretation of each state, we follow Jotikasthira et al. (2015) by projecting the foreign variables on the associated U.S. variables, and taking the residual as the foreign country specific states. This projection is also consis-

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2See Anderson et al. (2010) for a similar modeling strategy.
tent with the literature of international transmission of shocks, which mainly finds that the
U.S. market has dominant role in driving global financial market.

For home country, we denote \( X_t = [P_t', M_t']' \) and the Forex factor as \( x_t \), where \( P_t \) includes the first two principal components of bond yield curve and \( M_t \) includes the measures of inflation and growth. We assume that \( X_t \) and \( x_t \) follow the process:

\[
X_{t+1} = \mu_o + \psi_o X_t + \Sigma_o \eta_{t+1},
\]

\( (1) \)

\[
x_{t+1} = \mu_x + \phi_x x_t + \sigma_x \nu_{t+1},
\]

\( (2) \)

where \( \mu_0 \) is \( 4 \times 1 \), \( \psi_o \) and \( \Sigma_o \) are \( 4 \times 4 \) matrices, \( \mu_x, \phi_x, \sigma_x \) are all scalars. For foreign country, the definitions of states are similar but with superscript *. The dynamics of those states are assumed to follow:

\[
X^*_{t+1} = \mu^*_o + \psi^*_o X_t + \psi^*_f X^*_t + \Sigma^*_{oh} \eta^*_{t+1} + \Sigma^*_{of} \eta^*_{t+1},
\]

\( (3) \)

\[
x^*_{t+1} = \mu^*_x + \phi^*_x x_t + \sigma^*_x \nu^*_{t+1},
\]

\( (4) \)

where \( \mu^*_o \) is \( 4 \times 1 \), \( \psi^*_o, \psi^*_f, \Sigma^*_{oh} \) and \( \Sigma^*_{of} \) are \( 4 \times 4 \) matrices. Above dynamics actually assume that the home country states affect the foreign states, whereas the opposite transmission is not allowed. Also, the common Forex factor is assumed not to affect the observable states of both countries. As mentioned before, the existence of such component is crucial for fitting the exchange rate data, and can be motivated as common long-run growth component in the equilibrium model of [Colacito and Croce (2011)].

2.2. Pricing kernel

Denote \( Z_t = [X_t, X^*_t, x_t] \) as the collection of all states of two countries. According to assumptions in the previous subsection, the dynamics of \( Z_t \) can be written as:

\[
Z_{t+1} = \mu + \Phi Z_t + \Sigma \epsilon_{t+1},
\]

\( (5) \)

where \( \mu = \begin{bmatrix} \mu_0 \\ \mu^*_o \\ \mu_x \end{bmatrix}, \Phi = \begin{bmatrix} \psi_0 & 0 & 0 \\ \psi^*_o & \psi^*_f & 0 \\ 0 & 0 & \phi_x \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_o & 0 & 0 \\ \Sigma^*_{oh} & \Sigma^*_{of} & 0 \\ 0 & 0 & \sigma_x \end{bmatrix}, \epsilon_t = \begin{bmatrix} \eta_t \\ \eta^*_{t} \\ \nu_t \end{bmatrix}. \)

3See e.g. [Ehrmann et al. (2011)].
Assume that the log domestic economy-wide nominal pricing kernel is given by:

\[ m_{t+1} = -r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_{t+1}', \] (6)

where \( r_t \) is the domestic short rate, \( \lambda_t \) consists of the time-varying prices of risk. Following the literature of affine term structure, \( \lambda_t \) is assumed to be affine in states:

\[ \lambda_t = \lambda_0 + \lambda_1 Z_t, \] (7)

where \( \lambda_0 \) is a \( 9 \times 1 \) and \( \lambda_1 \) is a \( 9 \times 9 \) matrix. To be consistent with the orthogonality of Forex factor, the off-diagonal elements of \( \lambda_1 \) that correspond to it are assumed to be zero. Combining this assumption with the physical dynamics of \( Z_t \), equation (6) can be written as:

\[ m_{t+1} = m^B_{t+1} - \frac{1}{2} \lambda^2_{xt} - \lambda_{xt} \nu_{t+1}, \] (8)

where \( m^B_{t+1} \) is the bond-specific pricing kernel, \( \lambda_{xt} = \lambda_{0x} + \lambda_{1x} x_t \) is the risk price of Forex factor.

### 2.3. Restrictions on risk premia parameters

The price of risk \( \lambda_t \) is determined by state vector \( Z_t = [P'_t, M'_t, P'^*_t, M'^*_t, x_t]' \) as well as the risk premia parameters \( \lambda_0 \) and \( \lambda_1 \), which contain a large number of free parameters. To avoid the over fitting, we therefore impose several restrictions on \( \lambda_0 \) and \( \lambda_1 \) of both countries.

To facilitate the presentation, \( \lambda_0, \lambda_1, \lambda_0^*, \lambda_1^* \) are written in the following block form:

\[
\lambda_0 = \begin{bmatrix}
\lambda_{0P} \\
\lambda_{0M} \\
\lambda_{0P^*} \\
\lambda_{0M^*} \\
\lambda_{0x}
\end{bmatrix}, \quad \lambda_1 = \begin{bmatrix}
\lambda_{PP} & \lambda_{PM} & \lambda_{P^*P} & \lambda_{PM^*} & \lambda_{P^*_x} \\
\lambda_{MP} & \lambda_{MM} & \lambda_{M^*P} & \lambda_{MM^*} & \lambda_{M^*_x} \\
\lambda_{P^*P} & \lambda_{P^*M} & \lambda_{P^*^*P} & \lambda_{P^*M^*} & \lambda_{P^*^*_x} \\
\lambda_{M^*P} & \lambda_{M^*M} & \lambda_{M^*^*P} & \lambda_{M^*^*M^*} & \lambda_{M^*_x} \\
\lambda_{xP} & \lambda_{xM} & \lambda_{x^*P} & \lambda_{xM^*} & \lambda_{xx}
\end{bmatrix}
\]

\[
\lambda_0^* = \begin{bmatrix}
\lambda_{0P}^* \\
\lambda_{0M}^* \\
\lambda_{0P^*}^* \\
\lambda_{0M^*}^* \\
\lambda_{0x}^*
\end{bmatrix}, \quad \lambda_1^* = \begin{bmatrix}
\lambda_{P^*P} & \lambda_{P^*M} & \lambda_{P^*^*P} & \lambda_{P^*M^*} & \lambda_{P^*^*_x} \\
\lambda_{M^*P} & \lambda_{M^*M} & \lambda_{M^*^*P} & \lambda_{M^*^*M^*} & \lambda_{M^*_x} \\
\lambda_{P^*P} & \lambda_{P^*M} & \lambda_{P^*^*P} & \lambda_{P^*M^*} & \lambda_{P^*^*_x} \\
\lambda_{M^*P} & \lambda_{M^*M} & \lambda_{M^*^*P} & \lambda_{M^*^*M^*} & \lambda_{M^*_x} \\
\lambda_{xP} & \lambda_{xM} & \lambda_{x^*P} & \lambda_{xM^*} & \lambda_{xx}
\end{bmatrix}
\]
The restrictions are made based on some evidence in the empirical literature. For home country investor, consistent with (1), we assume that there are no risk compensations for the foreign states. This exploits the empirical facts (see e.g. Eun and Shim (1989)) that the shocks from U.S. financial market have large impact on other markets, but not vice versa. The same empirical pattern also provides guidance on restricting the price of risk for foreign country, that is, the parameters in $\lambda_1^*$ remain unrestricted to reflect such transmission of shocks. In terms of $\lambda_0$ and $\lambda_0^*$, we restrict $\lambda_0P^*, \lambda_0M^*, \lambda_0^*P, \lambda_0^*M$ to be zero. Since $\lambda_0$ characterizes the long run mean of bond yield, this assumption simply claims that the long run mean of one country’s bond yield is only determined by its country-specific parameters.

2.4. Bond prices, exchange rate and the implied variance

Following the literature, we assume that the one-period short rate of home country is an affine function of the home level and slope factors, that is,

$$ r_t = \delta_0 + \delta_1 P_t. $$

(9)

Then the bond yield with maturity $n$ periods ahead admits an affine form:

$$ y_t = a_n + b_n' P_t, $$

(10)

where $a_n = -\frac{A_n}{n}, b_n = -\frac{B_n}{n}, A_n$ and $B_n$ follow the recursions:

$$ A_n = A_{n-1} + B_{n-1}' (\mu_p - \Sigma_P \lambda_0P) \frac{1}{2} B_{n-1}' \Sigma_P B_{n-1} + A_1, $$

(11)

$$ B_n' = B_{n-1}' (\Phi_P - \Sigma_P \lambda_1P) + B_1', $$

(12)

where $\mu_P, \Sigma_P, \Phi_P$ are sub-matrix of $\mu, \Phi, \Sigma$ in (5), and $\lambda_0P, \lambda_1P$ are sub-matrix of $\lambda_0, \lambda_1$ in (7) that correspond to pricing factors $P_t$. The derivation of foreign bond prices is similar and thus omitted here.

For quantities related to the exchange rate, we assume that both home and foreign markets are complete, then the log nominal exchange rate return is the difference between log SDF (see Backus et al. (2001)) :

$$ \Delta s_{t+1} = m_{t+1}^* - m_{t+1} = (r_t - r_t^*) + \frac{1}{2} (\lambda_t' \lambda_t - \lambda_t^* \lambda_t^*) + (\lambda_t - \lambda_t^*) \epsilon_{t+1}. $$

(13)

### Footnote

4 After applying those restrictions, $\lambda_0P^*, \lambda_0M^*, \lambda_P^*, \lambda_M^*, \lambda_P^*, \lambda_M^*, \lambda_P, \lambda_M, \lambda_P^*, \lambda_M^*, \lambda_P^*, \lambda_M^*$ will be blocks of zero.
In this paper, we also explore the implications for currency option implied volatility. Previous literature of affine term structure model only focuses on the level of bond yield or exchange rate return, nonetheless, the model also has testable implications for the implied volatility. In particular, we show in Appendix B that a closed form expression for the risk neutral one-period ahead expectation of conditional variance $E_t^Q[\sigma_{t+1}^2]$ can be obtained within this model. Hence we could include the data of currency implied variance in the model estimation. As suggested by Graveline (2006), the option implied volatility provides useful information about the exchange rate volatility that is much harder to identify from the time-series data on exchange rate.

2.5. Complete market and exchange rates

Before proceeding to the solution method of the model, it’s worthwhile to discuss an important aspect of the model setting. As pointed out by Backus et al. (2001), equation (13) is the sufficient and necessary condition for the determination of exchange rate when no arbitrage holds and markets in both countries are complete (such that the comprehensive stochastic discount factor in both countries are unique). However, in most affine term structure models, the SDF is identified through bond prices. This identification though provides good fit of bond yield curve, by construction it has difficulty in accounting for exchange rate data since volatility of exchange rate return is much higher than that of interest rate. Therefore, equation (13) is easily rejected by the data. Previous studies then try different ways to deal with the term structure and exchange rate in a unifying framework. One of an early example is in Brandt and Santa-Clara (2002), where they abandon the complete market setting and use an exogenous process $o_t$ to bridge the gap between exchange rate and ratio of SDF (by setting $\Delta s_t = m^*_t - m_t + o_t$). Nonetheless, this procedure is ad hoc and more importantly, equation (13) now becomes necessary but not sufficient condition for exchange rate determination. This will create a theoretical drawback for the determination of exchange rate.

In contrast, the assumption of complete market setting is preserved in this model. Notice that the relation (13) shall hold with the comprehensive stochastic discount factor that price all asset payoffs, it’s then feasible to construct pricing kernel such that a part of it is used to pricing bonds, while the rest can only be identified through other asset classes such as exchange rate.

This motivates the use of Forex factor, not directly on the exchange rate relation (13) as in Brandt and Santa-Clara (2002), but as the state of SDF. The orthogonality of such factor with respect to bond states gives the model flexibility of fitting exchange rate

\footnote{See a similar argument in Joslin et al. (2014).}
data, while not deteriorating its bond pricing performance.

3. Data and Econometric Methodology

3.1. Data

We sample the data at monthly frequency from 1996M5-2016M2. We download U.S. nominal bond yields from Fed H.15 release\(^6\) and U.K. nominal bond yields from the Bank of England. We consider the maturities with 0.25, 0.5, 1, 2, 3, 5, 7, 10 years for U.S. and 1, 2, 3, 5, 7, 10 years for U.K.. We take one-month interbank rate as a proxy for short rate\(^7\) and the data is downloaded from Global Financial Database. For macroeconomic states, year over year (YOY) industrial production growth and CPI growth are treated as proxies for economic growth and inflation respectively. The exchange rate return is the log growth of spot exchange rate. Implied variance is calculated from at-the-money one month European currency options as in \textit{Londono and Zhou} (2014). The data is obtained from Bloomberg. The interest rate and implied variance data are denominated in the annual frequency, we transform them to the monthly frequency by dividing 12. For daily data, we use the data on the last trading day of each month to form the monthly sample. When discussing the implications for stock market, we will use the stock return and predictors’ data from Amit Goyal’s website. The details are provided in Appendix A.

3.2. Solution method

The model to be estimated consists of the following measurement equations:

\[
y^n_t = a_n + b'_n P^*_t + \zeta_{1t}, \tag{14a}
\]
\[
y^{*n}_t = a^*_{n} + b^{'*}_n P^{*t}_t + \zeta_{2t}, \tag{14b}
\]
\[
\Delta s_t = \hat{\Delta}s_t + \zeta_{3t}, \tag{14c}
\]
\[
E^Q_t[\sigma^2_{t+1}] = E^Q_t[\hat{\sigma}^2_{t+1}] + \zeta_{4t}, \tag{14d}
\]

where the symbols with hat are model-implied quantities, \(\zeta_{1t}, \zeta_{2t}, \zeta_{3t}\) and \(\zeta_{4t}\) are measurement errors of the data. (14a)-(14b) are the equations for the bond yields, (14c)-(14d) are the equations for the exchange rate determination and currency implied variance. The Forex

\(^6\)https://www.federalreserve.gov/releases/h15/data.htm

\(^7\)Following a large literature such as \textit{Verdelhan} (2016), we use forward discount as an alternative measure of short rate difference.
factor dynamics is the state transition equation:

\[ x_{t+1} = \mu_x + \phi_x x_t + \sigma_x \epsilon_{t+1}. \]  

\[ (15) \]

System (14)-(15) constitutes a nonlinear state space model and can be estimated via maximum likelihood. However, empirically estimating such a model is challenging for two reasons. First, the number of parameters is large. Second, the likelihood based estimation calls for unscented kalman filtering or particle filtering. Whether those methods can perform well for the parameter estimation, especially when the number of parameters is so large, is questionable. Consequently, we use a feasible two-stage estimation scheme by exploiting the orthogonal structure of Forex factor with respect to bond pricing.\(^8\) In the first stage, we estimate the affine term structure model (14a)-(14b). Then in the second stage, we estimate a nonlinear state space formed by (14c)-(14d), by fixing the point estimates obtained from the first step. More specifically, at the first stage, we estimate the following linear Gaussian state space model:

\[ y^n_t = a_n + b_n^P P_t + \zeta_{1t}, \]  

\[ (16a) \]

\[ y^*_{nt} = a_n^* + b_n^P P_{nt}^* + \zeta_{2t}, \]  

\[ (16b) \]

\[ V_t = X_t + \zeta_{1t}' \]  

\[ (16c) \]

\[ V_{t}^* = X_{t}^* + \zeta_{2t}' \]  

\[ (16d) \]

\[ X_t = \mu + \Phi X_{t-1} + \Sigma \epsilon_t, \]  

\[ (16e) \]

\[ X_{t}^* = \mu^* + \Phi^* X_{t-1}^* + \Sigma^* \epsilon_{t}^*, \]  

\[ (16f) \]

where measurement equations are (16a)-(16d), and state equations are formed by (16e)-(16f). The measurement errors are assumed as: \( \zeta_{1t} \sim N(0, \sigma_h^2), \zeta_{2t} \sim N(0, \sigma_f^2), \zeta_{1t}' \sim N(0, \sigma_m^2), \zeta_{2t}' \sim N(0, \sigma_m^2). \) In other words, bond yields and macroeconomic states of both countries are observed with errors. The standard deviations for measurement errors of domestic and foreign bond yields are identical within each country but different across countries, while the measurement errors for macroeconomic states of both countries follow exactly the same distribution.

System (16) can be estimated using maximum likelihood, where likelihood is evaluated via Kalman filtering. The number of parameters at this stage is still quite large, thus a good starting value is needed. We use a linear estimator proposed by de Los Rios (2015) as the starting value for optimization. The advantage of such method lies in its simplicity and robustness, as well as delivering sensible parameters that help fit the data.

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\(^8\)Similar multi-step estimation strategy can be found in Baele et al. (2010) and Corradi et al. (2013).
In the second step, we estimate the following nonlinear state space model:

\[
\Delta s_t = \Delta \hat{s}_t + \zeta_{3t}, \quad (17a)
\]
\[
E_t^Q[\sigma_{t+1}^2] = E_t^Q[\sigma_{t+1}^2] + \zeta_{4t}, \quad (17b)
\]
\[
x_t = \mu_x + \phi_x x_{t-1} + \sigma_x \epsilon_t, \quad (17c)
\]

where (17a)-(17b) are measurement equations, and (17c) is the state equation. Here the error terms \(\zeta_{3t}\) and \(\zeta_{4t}\) follow scaled t-distribution with scales \(\sigma_3, \sigma_4\) and degree of freedom \(\nu_3, \nu_4\) respectively. [Jacquier et al. (2004)] show that the model with t-distribution error term is more flexible in dealing with outliers. Due to the extreme observations during the period of financial turmoil in 2008, t-distributed measurement error is more suitable than standard normal distribution in this context. Moreover, we impose the restriction on \(\sigma_3\) and \(\sigma_4\) such that the model explains most of variations in the data, and the measurement error can account up to 10% of total fluctuations.\(^9\)

The number of parameters at this stage is medium and manageable. We estimate the parameters and the latent state in this system jointly using auxiliary particle filtering, which gives an unbiased estimate of likelihood and thus is a quite popular method in estimating the parameters of nonlinear state space. For the initial values of optimization, we consider many sets of initial parameters, which are drawn randomly from a reasonable domain, then we run maximum likelihood and only keep the estimates with the largest likelihood in the end.

At this stage, it is worthwhile pointing out the econometric role played by the measurement equation of implied variance (17b). Since in the second step estimation, only exchange rate related quantities are used, and notice that the Forex factor is assumed to reconcile the fluctuations of exchange rate return, with no impact on bond pricing. Thus the model is exactly-identified if we ignore (17b), and the model may potentially fit the data arbitrarily well (even though (17a) is a highly nonlinear function of Forex factor). After introducing (17b) as one additional measurement equation, the system is now over-identified and the model fit is not perfect ex ante.

\(^9\)This assumption is also made during the model estimation in [Schmitt-Grohé and Uribe (2012)].
4. Estimation Results

4.1. Results of the first-stage estimation

Table 1 displays the VAR estimation results of observable states for both countries as well as the common Forex factor. The estimates of the diagonal of Φ show that all macro and Forex states are persistent. For diagonals of Σ, consistent with the intuition, the Forex factor, which is designed to describe the foreign exchange market, is far more volatile than other macro states. While most of the parameters on the off-diagonal of Φ are statistically insignificant, we can still obtain several interesting economic observations. First, the slope factor predicts positively almost all other states in either home or foreign country. This is consistent with previous literature documenting the strong (positive) predictive ability of yield curve slope on future economic activity (see e.g. Estrella (2005)). Second, the yield curve level predicts higher inflation in both countries. As shown by Diebold et al. (2006), the yield curve level factor can be treated as the bond market perception of long run inflation. In terms of cross-country transmission, an interesting pattern is that the increase of U.S. level triggers negative response of all U.K. states. Such an effect can be reconciled with the literature on international transmission of shocks. For example, Muntaz and Surico (2009) find that U.K. inflation positively respond to lower interest rate of other industrialized countries in a factor-augmented VAR framework. Given the dominant role of U.S. among the industrialized countries, it’s then natural to have higher interest rate, slope, inflation and growth for U.K. after a negative innovation in U.S. interest rate.

Turning to the price of risk, Table 2 shows the estimation result. The risk loadings for each country’s level and slope risk are similar. On the one hand, level risk is negatively affected by the level and slope factor, although the loadings are not significant for U.S. investor. On the other hand, the time-varying slope risk is negatively driven by the slope factor itself. This stems from the fact that a higher yield curve slope predicts the economic boom, during which the risk premia will be low.

When it comes to the parameters of risk price for inflation and growth, it’s clear that their magnitudes are much higher than those of bond pricing factors. This is not surprising given the fact that those parameters are identified from exchange rate, which is much volatile than bond yields. For inflation risk, higher inflation and economic growth will induce higher inflation risk premium in home country, while the effect in foreign country is opposite. This

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10 In classical affine term structure such as Joslin et al. (2014) those parameters can’t be identified solely from the bond market. They also propose the possible identification from other asset markets.
is consistent with a comparative study of U.S. and Euro area by Hördahl and Tristani (2010), in which they find that the inflation risk premium is lower when output gap is increasing for Euro area, but such relation is opposite for U.S.. The former is in line with the common wisdom of countercyclical risk premium, while the latter effect can emerge since there will be higher risk of inflation surprises at economic boom. Whether one state moves up or down the inflation risk premium depends on the relative magnitude of those two effects. This mechanism also explains why inflation risk premium of foreign country is positively and significantly driven by both country’s slope factor: a better prospect of economy can still increase the inflation risk premium when the latter effect dominates. For fluctuations in growth risk premium, the growth factor itself has the largest and significant effect for U.S., while the estimates for U.K. are insignificant. The opposite impact of economic growth factor on growth risk premium for two countries can be understood in analogy to the inflation risk premium.

The above discussions indicate that those two economies have heterogeneous responses to macro states. However, they have similar exposure to the Forex factor. Time-varying risk compensation for Forex factor is almost identical, with highly significant estimates. This justifies our assumption about its commonality between two countries, and also suggests that such factor may potentially capture some systematic risk.

[Insert Table 2 near here]

Though the focus of this paper is the fit of exchange rate data, the result can be misleading if the fit for bond worsens substantially, given the tight relation of cross-country SDF in equation (13). Thus we report the performance of yield curve fitting in Table 3. As can be seen from the table, the pricing errors are around 10 basis points (annulized) for all maturities. The magnitudes are small and comparable with studies that focus only on yield curve fitting (see e.g. Piazzesi (2005)), suggesting that the model has satisfactory description of bond yield data.

[Insert Table 3 near here]

4.2. Fit of option implied variance

In this subsection, we discuss the model fit for currency option implied variance. In the data, the mean and standard deviation of the (monthly) implied variance of one-month at-the-money option is 0.07% and 0.06%, while the model generated data has mean 0.06%
and standard deviation 0.02%. The measurement error with t-distribution has an estimated
degree of freedom 4.22, therefore indicating the fat tailedness in the option data.

The less volatile model-generated quantity is due to the asset volatility spike during 2008
financial crisis, when the macroeconomic factors do not display such dramatic movements.
Indeed, we show that after excluding the data points from 2007M12 to 2009M6, the standard
deviation for the data and model is now 0.02% and 0.01% respectively. The distance between
those two becomes much smaller compared to that obtained from the full sample. Interest-
ingly, the data points during the period of financial turmoil will push up the volatility of data
by 3 folds (from 0.02% to 0.06%), while the prediction from the model also doubles. Thus
macroeconomic fundamentals and implied volatility can display (conditional) co-movement,
instead of little connections as documented by Mixon (2002).

To formally evaluate the fit, we regress the data on the model-implied counterpart to
better see the relation between those two quantities, the regression results are:

\[ E_t^Q (\sigma_{t+1}^2) = -0.0006 + 2.045 \hat{E}_t^Q (\sigma_{t+1}^2) + \epsilon_t, \quad R^2 = 52.3\%. \]

The coefficient on \( \hat{E}_t^Q (\sigma_{t+1}^2) \) is 2.05, which is significant with a t-statistic 16.01. The
deviation from unity regression coefficient is consistent with the above moment comparisons
of data and model. Also from the \( R^2 \) of the regression, a substantial proportion of variations
in the data can be explained by the model. The results thus suggest that the macro-factor
affine model, in addition to providing good fit for bond yield, also has the potential to track
the movements of currency option implied variance.

4.3. Fit of exchange rate return

Figure 1 displays the fit of exchange rate return. The model implied quantity tracks the
data quite well, with the correlation coefficient of 0.81. In particular, the model fit is almost
perfect except the periods of financial crisis. Recent papers including Brunnermeier et al.
(2009) and Adrian et al. (2015) find that extreme exchange rate movements may be related
to the funding liquidity. Since the macro states considered in this paper do not include
any liquidity-related measures, the bad fit during the crisis may partially be attributed to
omitted state variables.

[Insert Figure 1 near here]

To shed more lights on the underlying drivers of the model fit and the importance of
the Forex factor, we implement an exercise similar to variance decomposition. Formally, we
calculate the volatility of the return data and the model-implied return based on the measure proposed in Corradi et al. (2013):

\[ Vol_t = \sqrt{6\pi} \frac{1}{12} \sum_{i=1}^{12} |r_{t+1-i}|. \]  

(18)

Then we fix the Forex factor at its unconditional mean, and repeat the volatility calculations again. The difference between those two series of volatilities provides a measure of the importance of Forex factor for fitting the data. Similarly, to clarify the role of other macroeconomic states, we present the root-mean-square error (RMSE) between data and model implications when we shut off each macro state once at a time. The decomposition result is shown in Figure 2 and the RMSE measures are presented in Table 4.

[Insert Figure 2 near here]

[Insert Table 4 near here]

Obviously, a single Forex state can explain a substantial portion of changes in exchange rate volatility. This is not inconsistent with the well-known exchange rate disconnect puzzle stating that the short-term link between exchange rate and economic fundamentals is weak, since the Forex factor is latent and therefore does not map directly to fundamentals. Interestingly, we find that the pure macroeconomic and interest rate factors can still explain some part of the data. In particular, as can be seen from Table 4 inflation measures in two countries strongly affect the exchange rate volatility, since the model fit worsens substantially after we ignore the variation of U.S. or U.K. inflations. Many previous empirical studies on the macroeconomic determinants of exchange rate mainly focus on the predictive ability of fundamentals in the regression framework and find quite limited role for the macro factors (see a comprehensive review by Rossi (2013)). In contrast, the successful detection of the close relation between exchange rate and fundamentals in this paper is of interest for two reasons. First, the two-stage estimation scheme does not allow the dynamics of macroeconomic states to reconcile the exchange rate data, therefore the role of macroeconomic states is not a result of manipulation. Also, we implement the bottom up modeling strategy by starting from a reduced form model of stochastic discount factor and the no-arbitrage condition, then the exchange rate is determined from the cross-country difference in log stochastic discount factor, thus exchange rate is connected to the fundamentals in a more rigorous way compared to the regression method.

Another interesting observation arises by comparing last two columns of Table 4. Although the Forex factor is quite important for exchange rate return, it contributes almost
nothing to the fit of the option implied variance. This suggests that different drivers underlie exchange rate and option market. For currency options, the U.S. growth appears to be the most important state. This is intuitive because U.S. growth characterizes to a large extent the world economy prospects and thus the forward-looking nature of option market will treat it as an important source of risk.\footnote{Even though the implied volatility used here is at one month horizon, on which macroeconomic fundamentals may have quite limited impact directly, the strong co-movement of short-term and long-term implied volatility will transmit the macro effect on long-horizon options to short-horizon.}

5. Implications for Stock Markets

5.1. Results of predictive regressions

In this section, we explore the potential role of Forex factor, which according to previous discussions is an important factor entering the log SDF and determining the exchange rate volatility, for the stock markets of U.S. and U.K.. For the simplicity of notations, we denote the value of such factor at time $t$ as $FX_t$.

Following the literature on stock return predictability (see recent papers e.g. Welch and Goyal (2008), Rapach et al. (2016)), we study the following predictive regression:

$$r_{t+1} = \alpha + \beta z_t + \epsilon_{t+1},$$

where $r_{t+1}$ is the one-period ahead excess return, and the predictor $z_t$ characterizes the time-varying expected return. We utilize (19) to test the ability of Forex factor in capturing the time-varying expected return of stock market. For comparison, we also evaluate the performance of other 14 predictors that are commonly used in the literature. Since data of all corresponding predictors in U.K. is not available, we use U.S. predictors to forecast the U.K. stock market. The details of constructing those variables can be found in the Appendix A. Table 5 gives the predictability results for the excess returns of aggregate stock market of both countries.

\[\text{Insert Table 5 near here}\]

A clear result from Table 5 is that most predictors have negligible ability in forecasting the excess return. If risk premium is indeed time-varying, then the weak predictive relation is either due to wrongly selected variables, or lack of predictability during specific periods as discussed in Welch and Goyal (2008).\footnote{They find that for the most recent 20 years up to their work, even the in-sample predictability is very poor.} Strikingly, the Forex factor, which is constructed...
solely from the foreign exchange market, significantly predict stock market risk premium of two countries with the Newey-West t-statistics 2.72 and 2.28 respectively. The sign of slope coefficient is also consistent with the estimates of $\lambda_{1x}^*, \lambda_{1x}^*$ in Table 2. Higher $FX_t$ will lower the aggregate risk premium in both countries.

A potential explanation for the commonality in the foreign exchange and stock markets within our sample periods may be due to the crisis. It’s well known that the correlations among different asset classes will peak during the crisis periods, it’s therefore worthwhile to investigate to what extent the predictability can be attributed to the comovements during the recession. More specifically, we run a predictive regression by controlling for a dummy variable that takes the value of 1 during the NBER recession periods. In addition, we control for other predictors which may capture different risk of stock markets and report the incremental power of predicting the equity premium on top of each predictor by $FX_t$ in Table 6. The strong predictability by the Forex factor remains untouched after controlling for all other predictors including the recession dummy. Given the importance of the Forex factor in accounting for exchange rate fluctuations, such remarkable performance from forecasting the aggregate stock return indicates that there exists common systematic risk between those two markets.

[Insert Table 6 near here]

Intuitively, if Forex factor indeed well captures the systematic risk, it shall also forecast risk premia of assets that may have different risk exposure. We thus use Forex factor to forecast returns of a variety of characteristic portfolios. The results are presented in Figure 3. The Forex factor significantly predicts most of sorted portfolios, without significant cross-sectional pattern of loadings on the predictor. Noticeably, the estimated slope coefficients are also negative, same with that of the market risk premium. In all, the forecasting exercise implies that the foreign exchange market provides important information about risk-return relation in the stock markets, and such information is orthogonal to commonly perceived stock market risk factors.

[Insert Figure 3 near here]

5.2. Long-horizon predictability

This subsection evaluates whether the Forex factor has forecasting power over longer horizons. Since Forex-specific factor enters into the pricing kernel, it shall not only has

\[^{13}\text{Within our sample period, the NBER business cycle dates are March 2000 to November 2001 and December 2007 to June 2009.}\]
predictive power on returns of different assets (as shown in the previous subsection), but also predicts the multi-period risk premium of a single asset. Therefore we run the long-horizon predictive regression on market return under different holding periods to test the long-run predictability.

Table 7 reports the in-sample forecasting results for stock market excess returns of two countries. The table shows that the Forex factor can consistently predict the short and long run market risk premium, with horizons ranging from one month to three years. The short term predictability peaks at around one-quarter horizon, while that for long term (over one year) peaks at three-year horizon. For S&P 500 index, the Forex factor explains almost 10% variance of the one quarter ahead cumulative excess return, while for FTSE index, such factor has relatively less predictive power but still with noteworthy adjusted $R^2$ 5.94%. Even at three-year horizon, the adjusted $R^2$ can reach 8.01% and 12.5% for both markets.

[Boudoukh et al. (2008)] finds that the slope coefficients of long and short horizon predictive regression are highly correlated if the predictor is persistent, thus result of long run regression does not provide much new insight beyond that of short run predictive regression. They show that the correlation between the slope coefficient of one-period and that of $k$-period predictive regression is:

$$
(1 - \rho)^2 + \rho(1 - \rho)(1 - \rho^{k-1})
\over(1 - \rho)\sqrt{k(1 - \rho)^2 + 2\rho[(k - 1) - \rho(k - \rho^{k-1})]}.
$$

The estimated persistence of Forex factor is 0.82, therefore equation (20) implies that the correlations between one-month and 1,2,3 year slope coefficients are 0.58, 0.40 and 0.31, respectively. In particular, the coefficient for three-year horizon is moderately correlated with that of one month horizon, yet both are significant at 5% level, with the Newey-West t-statistics 2.72 and 2.04 for U.S. and 2.28 and 2.23 for U.K.. Those results suggest that the long horizon slope coefficient is not a repetition of short run estimates, but instead uncovers the predictive power of the Forex factor for stock markets at the long run.

5.3. Cross-section of industry portfolios

Francis et al. (2008) finds that even though currency risk is significantly priced at the aggregate market, it’s puzzling why such risk does not show up at the industry level since there are numerous studies on the exchange rate effects on industries. Having discussed the impact of Forex factor on aggregate stock market from the time-series prospective, it’s
then of interest to see its explanatory power for the cross-sectional return differences in the industry portfolios.

More specifically, we examine the following model using 30 industry portfolios as testing assets:

\[ E[R^i] = \alpha^i + \beta^i_M \lambda_{MKT} + \beta^i_{SMB} \lambda_{SMB} + \beta^i_{HML} \lambda_{HML} + \beta^i_{RMW} \lambda_{RMW} + \beta^i_{CMA} \lambda_{CMA} + \beta^i_{FX} \lambda_{FX}, \]

(21)

where \( R^i \) is the excess return of \( i \)-th industry portfolio, and \( MKT, SMB, HML, RMW, CMA \) represent the five factors recently proposed by Fama and French (2015). We use traditional cross-sectional regression approach for estimating (21). In the first step, we estimate the factor betas by running the time-series regression of excess returns on six factors. In the second step, we estimate the cross-sectional regression using mean excess return and the factor betas obtained from the first step. Note that controlling for other commonly used factors helps pin down the importance of Forex factor for explaining the cross-sectional differences of industry portfolio returns. For comparison, we also report the results when only market factor or Fama-French three factors are used respectively.

Table 8 presents the estimation results. First, the classical CAPM or Fama-French three-factor model can’t explain the differences in average returns of industry portfolios. The adjusted \( R^2 \) of cross-sectional regression is low. This is consistent with estimates in Lewellen et al. (2010), who find that adding 30 industry portfolios as testing assets will substantially deteriorate the pricing ability of many celebrated models. A striking result emerges when we add Forex factor as a cross-sectional risk factor. Even the original CAPM model now has much larger explanatory power for cross-section return dispersions, with adjusted \( R^2 \) jumping from 4.3% to 41.9%, while the incremental power for Fama-French three-factor model is also remarkable, from 21.1% to 39.7%. Another observation is that the most recent Fama-French five-factor model performs well and increases the adjusted \( R^2 \) by twofold compared to that of the three-factor model. Fama and French (2016) find that adding profitability (RMW) and investment (CMA) factors help alleviate several well-known cross-section average return anomalies. The results here suggest that those two additional factors are also useful for explaining cross-industry return differences. Interestingly, the classical CAPM augmented with the Forex factor has almost the same explanatory power with the five-factor model, this indicates that the Forex factor is also an important risk factors at the cross-section dimension. It should be noticed that most alphas remain significant after including different risk factors, which highlights the difficulty of explaining the return differences of those portfolios. Yet our

\[ \text{Data of industry portfolios are available from Kenneth French’s website.} \]
focus here is to document the relative importance of Forex factor, which may not necessarily capture all the important risk in the industry portfolios.

From the estimation of full specification (21), we find that the predictive power of FX factor is partially subsumed in the CMA factor, yet it still provides additional information since the adjusted \( R^2 \) rises from 42.7% to 45.1%. The fact that the investment factor and the Forex factor contain common information about stock returns suggests that the Forex factor may affect the cash-flow part of stock prices since common measures used to construct investment factor (e.g. the total asset growth as in Fama and French (2015)) is closely related to firms’s cash flow. Indeed as we will show in the next subsection, by decomposing the return into the discount rate and cash flow parts, the Forex factor strongly forecasts the cash-flow news of stock returns.

5.4. Return decomposition

To gain more insights about the origin of predictability by the Forex factor, we work with Campbell and Shiller (1988) decomposition:\(^{15}\)

\[
r_{t+1} - E_t r_{t+1} = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta r_{t+1+j},
\]

where \( \rho = \frac{1}{1+\exp(dp)} \). Denote unexpected return, cash flow news and discount rate news as \( \eta_r, \eta_{CF}, \eta_{DR} \), then equation (22) becomes:

\[
\eta_r = \eta_{CF} - \eta_{DR}.
\]

Following Rapach et al. (2016) and a large literature, the news component can be extracted from a VAR model with state \( X_t = [r_t, dp_t, z_t] \):

\[
X_{t+1} = \mu + AX_t + u_{t+1},
\]

where \( r_t \) is the market excess return, \( dp_t \) is the log dividend-price ratio, \( z_t \) is some additional state characterizing the economy beyond dividend-price ratio.\(^{16}\) Denote the 0-1 selection vector \( e_1 \), whose elements are all zero except the position that corresponds to market excess

\(^{15}\)Due to lack of U.K. stock market data, we only implement such decomposition for U.S. market.

\(^{16}\)Following Rapach et al. (2016), we choose \( z_t \) from predictors that appear in Welch and Goyal (2008).
We identify the discount rate news from VAR model (24) directly, and take the residual as the cash flow news. In other words, the news component can be estimated as:

\[
\begin{align*}
\eta_{t+1}^r &= \epsilon_1' u_{t+1}, \\
\eta_{t+1}^{DR} &= \epsilon_1' \rho A(I - \rho A)^{-1} u_{t+1}, \\
\eta_{t+1}^{CF} &= \eta_{t+1}^{DR} + \eta_{t+1}^r, \\
\end{align*}
\]

Then we estimate the following regressions after obtaining the news component:

\[
\begin{align*}
E_{t} r_{t+1} &= \alpha_E + \beta_E F X_t + \epsilon_{t+1}^E, \\
\eta_{t+1}^{CF} &= \beta_{CF} F X_t + \epsilon_{t+1}^{CF}, \\
\eta_{t+1}^{DR} &= \beta_{DR} F X_t + \epsilon_{t+1}^{DR}, \\
\end{align*}
\]

where \(E_{t} r_{t+1} = \epsilon_1' (\mu + AX_t)\).

Comparing the magnitude of slopes in system (25), it’s then straightforward to check the source of predictability. More specifically, \(\beta_E, \beta_{CF}, \beta_{DR}\) characterize the predictive ability of the Forex factor on the expected return, cash flow news and discount rate news. We report the regression results in Table 9. The Forex factor has a strong predictive ability on the news of future cash flows, which is robust across different conditioning variables \(z_t\).

Even though FX factor is modeled as a driving force of stochastic discount factor, the ability of forecasting news of future cash flows is not inconsistent with the model. In many asset pricing models where investor has either CRRA or the Epstein-Zin preference, the log SDF is connected to the consumption growth, whose variation is clearly linked to that of future cash flows of contingent claims. In addition, the predictability of cash flow news is in line with the findings documented by Atanasov and Nitschka (2015). In an ICAPM framework, they show that a common source of systematic risk in stock and currency returns is reflected in the market return’s cash-flow news.

Simultaneously, the FX factor also predicts significantly the news of discount rate for many conditioning variables. Conforming to the intuition, the predictive relation is opposite for cash-flow news and discount-rate news, meaning that a positive cash-flow news will be accompanied by a negative discount rate news, or a lower risk premium.

[Insert Table 9 near here]

\[\text{Chen and Zhao (2009) show that the identification scheme may be problematic if there are any misspecifications in the predictability. It will affect the estimated discount rate news directly and cash flow news indirectly. Following a remedy by Maio and Philip (2015), we repeat the exercise by first identifying the cash flow news, and treat the rest as discount rate news. The results are quantitatively similar.}\]
5.5. Why does the Forex factor forecast the stock risk premium?

As we can see from (6), the Forex factor enters into the pricing kernels in a nonlinear manner. Therefore even though the return predictability is a popular exercise of examining the risk factors, it’s worthwhile to investigate why the Forex factor forecasts the market risk premium in a linear way. This will be important both for uncovering the key mechanism and for keeping the results from the concern of statistical artifacts.

Here we argue that the Forex factor strongly captures the movement of the conditional variance of log SDF, $Var_t[m_{t+1}]$, which in equilibrium is an (infeasible) predictor of aggregate risk premium. More specifically, we plot the standardized conditional variance of log SDF and Forex factor in Figure 4. The figure shows that the variance of log SDF are almost entirely driven by the Forex factor, the correlation reaches 0.99. The dominant role can be attributed to the higher volatility of estimated Forex factor comparing to those of observable states.

The role of the conditional variance of log SDF can best be seen from a simple textbook economy where agent has the Epstein-Zin preference and the log return and log SDF are jointly normal. For example, consider the asset pricing equation for return of claim to aggregate consumption $R_{t+1}^c$:

$$E_t[M_{t+1}R_{t+1}^c] = 1. \tag{26}$$

Assume that $M_{t+1}$ and $R_{t+1}^c$ are conditionally log-normal, then above equation reduces to:

$$E_t(r_{t+1}) - r_f^t = -\frac{1}{2} Cov_t(m_{t+1}, r_{t+1}^c) - \frac{1}{2} Var_t(r_{t+1}^c), \tag{27}$$

where small letters denote the logarithm quantities. When investor has Epstein-Zin preference:

$$U_t = \{(1 - \beta)C_t^{\frac{1-\gamma}{\psi}} + \beta(E_tU_{t+1}^{1-\gamma})^{\frac{1}{\psi}}\}^{\frac{\theta}{1-\gamma}}, \tag{28}$$

where $\theta = \frac{1-\gamma}{1-\psi}$, the log SDF can be written as:

$$m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)r_{c,t+1}. \tag{29}$$
Combining (29) and (27), we thus have:

\[ E_t(r_{t+1}) - r_t = -\frac{1}{2(\theta - 1)} Var_t(m_{t+1}) - \frac{\theta}{2\psi(\theta - 1)} Cov_t(m_{t+1}, g_{t+1}). \]  

(30)

From equation (30), it’s now clear that any factors driving the conditional variance of log SDF shall be good candidate for predicting risk premium.

[Insert Figure 4 near here]

However, it shall be noted that the reduced form stochastic discount factor (6) may not mimic the SDF in this simple economy, therefore above explanation is not consistent with the empirical findings in previous sections. Nevertheless, we show below that the evidence of predictability can still be reconciled using a more complicated equilibrium model.

To make the model tractable, we work within the continuous time framework. Notice that the continuous counterpart of (6) is:

\[ \frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t^t dW_t, \]  

(31)

where \( \lambda_t \) is given in (7). We consider the case with only one state variable \( Z_t = x_t \), which follows an OU process with zero mean:

\[ dx_t = -\kappa x_t dt + \sigma F dW_t, \]  

(32)

where \( F \) is \( 1 \times m \) vector, with \( FF' = 1 \). \( W_t \) is a \( m \)-dimensional Brownian Motion.

Define an auxiliary state \( y_t = x_t^2 \), the dynamics of \( y_t \) can be found by applying Itô’s lemma on (32):

\[ dy_t = (\sigma^2 - 2\kappa y_t) dt + 2\sigma y_t F dW_t. \]  

(33)

Assume that the process for consumption is:

\[ \frac{dC_t}{C_t} = \mu dt + \sqrt{y_t} L dW_t, \]  

(34)

where \( L \) is \( 1 \times m \) vector, with \( LL' = 1 \). Household has Epstein-Zin preference:

\[ J_t = \max_{\{C_t\}} E_t(\int_t^T f(C_s, J_s) ds), \]  

(35)
where the aggregator \( f(C, J) \) is given by:

\[
f(C, J) = \beta (1 - \gamma) \frac{C}{J^{1 - \psi}} \left(1 - \frac{1}{1 - \psi} \right) - 1.
\]  

(36)

Then we have the following proposition:

**Proposition.** Suppose the endowment economy is described as above, then the dynamics of equilibrium state price of density is:

\[
\frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t F dW_t.
\]  

(37)

The risk premium of consumption claim is:

\[
E_t \left( \frac{dP_t}{P_t} \right) + D_t \frac{dP_t}{P_t} dt - r_t dt = \frac{1 - \psi}{1 - \gamma} A_t \frac{\lambda^2 - \lambda_0 \lambda_1}{\lambda_1} K F' - \frac{\lambda^2 - \lambda_0 \lambda_1}{\lambda_1} KL',
\]  

(38)

where \( \lambda_t = \lambda_0 + \lambda_1 x_t \), \( F, K, L, A_1 \) are given in Appendix C.

**Proof.** See Appendix C. \(\square\)

The above proposition confirms that in this endowment economy, the reduced form SDF (6) is an equilibrium outcome and the property that stock risk premium is a linear function of conditional variance of log SDF (\( \lambda^2_t \)) is preserved. To the best of our knowledge, the equilibrium motivation for specification (6) has not been discussed before. Given the popularity of (6) in the literature of affine term structure model, it’s important to have a micro-founded explanation for that specification\(^{18}\). Also, according to this proposition, the predictability results presented in previous subsections should be an equilibrium regularity, once we have a correct model of stochastic discount factor.

### 6. Additional Implications and Robustness Checks

#### 6.1. Systematic risk factor from nonparametric method

A recent paper by Verdelhan (2016) confirms the existence of systematic risk in a number of bilateral exchange rates. He finds that the dollar factor, which is constructed as the average of all currency returns at each time period, is the main determinants of world-wide exchange

\(^{18}\) Bansal and Zhou (2002) also discusses an equilibrium explanation for exogenous specified pricing kernel, their method is to specify a process for \( r_{t+1} \) such that \( 29 \) coincides with \( 6 \). Here we use a bottom-up strategy by specifying a consumption process, and show that the equilibrium pricing kernel is of form \( 6 \).
rate fluctuations. It will be interesting to compare the dollar factor with the Forex factor obtained in this paper. Note that his method of extracting the factor is non-parametric, while ours relies on a fully parametrized SDF model. As can be seen from the scatter plot in Figure 5, those two factors identified using different methods are positively correlated with the correlation coefficient of 0.32. This again provides support for the interpretation of Forex factor as a systematic risk component. Moreover, Verdelhan (2016) shows that the Dollar factor depends on U.S.-specific shocks to pricing kernels, we therefore compare its ability on driving expected return of U.S. stock market with that of Forex factor over different forecasting horizons in the last two columns of Table 7. The results of the long-horizon predictive regression shows that the Dollar factor has almost no explanatory power for U.S. stock market. This highlights the usefulness of the parametrized SDF model in extracting important risk factors.

[Insert Figure 5 near here]

6.2. Forward premium anomaly

In this subsection, we explore how well the model can account for the forward premium anomaly as documented by Fama (1984). In the sample studied here, the UIP regression

$$\Delta s_{t+1} = \alpha + \beta (r_t - r_t^*) + \epsilon_{t+1}$$  \hspace{1cm} (39)

gives $\beta$ estimate of -1.62, with a t-statistic 0.93 and adjusted $R^2$ -0.06%. Theory indicates that if both currencies are equally risky, then investors expect the currency with high interest rates to depreciate, thus $\beta$ shall be positive. The deeply negative $\beta$ generates the so-called forward premium anomaly (or UIP puzzle). Fama (1984) attributes the anomaly to the existence of time-varying risk premium. If this is the case, then the risk-adjusted UIP regression shall be used instead of (39):

$$\Delta s_{t+1} = \alpha + \beta (r_t - r_t^*) + \gamma rp_t + \epsilon_{t+1}.$$ \hspace{1cm} (40)

Within the affine model, the risk premium term can be explicitly solved out:

$$rp_t = \frac{1}{2}(\lambda_t^t \lambda_t - \lambda_t^t \lambda_t^*)$$ \hspace{1cm} (41)

Estimating model (40) gives $\hat{\beta} = -0.72$ with a t-statistic 0.40, adjusted $R^2$ now increases to 0.88%. Though the model is insufficient to fully account for the anomaly, to some extent the
model implied risk premium does help alleviate it. The limited explanatory power for UIP puzzle is due to two reasons. First, one focus of this paper is to construct a parsimonious model that can track the dynamics of bond yield and exchange rates return. Thus our approach is different from Sarno et al. (2012) and Brennan and Xia (2006), who focus on resolving the forward premium anomaly and directly model the foreign exchange risk premium. In particular, we don’t impose restrictions on $\lambda_t$ and $\lambda^*_t$ to possibly reconcile the forward premium anomaly, instead we motivate the restrictions through the literature of international shock transmission. The second and more important reason is that the model implied risk premium is a deterministic function of observable states and the Forex factor. The former are exogenously given, while the latter is required to match the fluctuations of exchange rate return and option implied variance. Thus it’s not obvious that the risk premium identified in the model can also satisfy conditions proposed by Fama (1984) as necessary to explain the forward premium anomaly. Yet combing the less negative estimate of $\beta$ in (40) and the strong predictive ability on stock market risk premium of both countries by Forex factor, the model indeed captures important systematic risk.

6.3. Role of estimation errors

Since Forex factor is crucial in our analysis, and it is obtained through the particle filtering, it’s of great importance to ensure the accuracy of the estimation of latent state. We simulate 1000 sample paths using parameter estimates in Table 1 and 2, with the same length as the data sample. Then we implement the particle filters on the simulated data. We calculate the ratio of mean absolute error to the true state for each sample path to measure the difference between those two. The average filtering error from 1000 simulations is about 0.35%. To illustrate the accuracy of filtering more directly, we plot the true state and the filtered state from a randomly selected sample in Figure 6. The almost perfect match between the true and the estimated series demonstrate the accuracy of the particle filter and hence the estimation error of the latent state has limited impact on the results.

[Insert Figure 6 near here]

7. Conclusion

This article studies the share of systematic risk between the foreign exchange and the stock market through the lens of an affine term structure model. We treat the commonly used

\[19\text{Two conditions are: i) the implied risk premium is more volatile than, and ii) negatively correlated with the interest rate differentials.}\]
bond risk factors and one latent Forex factor as the states driving the stochastic discount factors of the two countries. The model has satisfactory fit for bond yields, exchange rate returns and currency option implied variance. The Forex factor turns out to be a strong predictor for aggregate stock market risk premium. In addition, it also greatly enhances the pricing ability of the classical CAPM and Fama-French three-factor model for cross-section industry portfolios, the performance is even comparable with that of the newly proposed Fama-French five factor model. The return decomposition finds that the Forex factor strongly predicts the cash-flow news of the aggregate market, and the cross sectional regression indicates that this factor shares the information with the investment factor. Therefore, the evidence from the time-series and the cross-section dimensions both points to the close relation between the Forex factor and the cash-flow, and echoes the theoretical results in Colacito and Croce (2011) that the common long-run growth in consumption is a key elements for resolving several asset pricing puzzles in the stock and the foreign exchange markets.

The results in this paper show that there is important information about economy-wide risk compensation in the foreign exchange market. It’s then of interest to see whether such information is important in many other asset markets and how it interacts with the macroeconomic fluctuations. Providing such analysis is beyond the scope of this article and therefore left to future research.

References


Table 1: **VAR estimation.** The table reports the estimation results of model (5): $Z_{t+1} = \mu + \Phi Z_t + \Sigma_{t+1}$, where $Z_t=$[U.S. Level, U.S. Slope, U.S. Inflation, U.S. Growth, U.K. Level, U.K. Slope, U.K. Inflation, U.K. Growth, Forex]. The empty cells in the table represent zero restrictions. Robust standard errors of [White (1982)] are reported in the parentheses. *, ** and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>$\mu$ ($\times 10^3$)</th>
<th>$\Phi$</th>
<th>$\Sigma$ ($\times 10^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.170</td>
<td>0.014</td>
<td>0.016</td>
</tr>
<tr>
<td>(0.223)</td>
<td>(0.001)'(0.060) (0.053) (0.020)</td>
<td>(0.034)'(0.058)'(0.018)'(0.045)</td>
</tr>
<tr>
<td>0.242</td>
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<td>0.912</td>
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<tr>
<td>(0.041)'(0.02) (0.001)'(0.032) (0.013)</td>
<td>(0.017)'(0.013)'(0.013)'(0.028)</td>
<td></td>
</tr>
<tr>
<td>0.071</td>
<td>0.005</td>
<td>0.013</td>
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<tr>
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<td>0.015</td>
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<tr>
<td>(0.013)</td>
<td>(0.035) (0.031)'(0.013)</td>
<td>(0.041) (0.032) (0.022)'(0.045)</td>
</tr>
<tr>
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<tr>
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<tr>
<td>(0.039)</td>
<td>(0.016) (0.113)'(0.098) (0.024)''</td>
<td>(0.071) (0.051) (0.112) (0.069)''</td>
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<tr>
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<tr>
<td>0.028</td>
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<td>0.028</td>
</tr>
<tr>
<td>(0.030)</td>
<td>(0.011) (0.057) (0.074) (0.038) (0.001)'(0.007) (0.031)'(0.051)</td>
<td>(0.034) (0.181) (0.091) (0.045) (0.215)* (0.092) (0.195) (0.108)</td>
</tr>
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<td>-0.069</td>
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<td>-0.010</td>
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<tr>
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<td>0.072</td>
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<tr>
<td>-0.003</td>
<td>-0.123</td>
<td></td>
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<tr>
<td>(0.053)</td>
<td>(0.008) (0.016) (0.075) (0.028) (0.007) (0.032)'(0.020) (0.020)</td>
<td>(0.029)'(0.053)'(0.125) (0.019) (0.055) (0.047)'(0.021) (0.050)''</td>
</tr>
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<tr>
<td>(0.164)</td>
<td>(0.012) (0.031) (0.019) (0.024) (0.075) (0.051)'(0.041)</td>
<td>(0.072) (0.063) (0.173) (0.059) (0.056) (0.199) (0.162) (0.248)</td>
</tr>
<tr>
<td>0.172</td>
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<td>0.008</td>
</tr>
<tr>
<td>-0.091</td>
<td>-0.046</td>
<td>-0.001</td>
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<tr>
<td>0.152</td>
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<td>0.860</td>
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<tr>
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<td>-0.017</td>
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<tr>
<td>0.028</td>
<td>-0.123</td>
<td>0.109</td>
</tr>
<tr>
<td>1.099</td>
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<td></td>
</tr>
<tr>
<td>(0.554)</td>
<td>(0.036) (0.110) (0.267) (0.056) (0.045) (0.172) (0.130) (0.085)***</td>
<td>(0.373) (0.417) (0.704) (0.097)'(0.468) (0.231) (0.441) (0.339)***</td>
</tr>
<tr>
<td>2.12</td>
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<tr>
<td>(9.40)</td>
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<tr>
<td>(0.047)***</td>
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<tr>
<td>6.292</td>
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</table>
Table 2: **Risk premia parameters.** The table reports the estimates of risk premia parameters. The four rows of $\lambda_0$ and four columns of $\lambda_1$ represent the parameters corresponding to U.S. Level, U.S. Slope, U.S. Inflation and U.S. Growth. The four rows of $\lambda_0^*$ represent the parameters for U.K. Level, U.K. Slope, U.K. Inflation and U.K. Growth, and eight columns of $\lambda_1^*$ represent those for U.S. Level, U.S. Slope, U.S. Inflation, U.S. Growth, U.K. Level, U.K. Slope, U.K. Inflation and U.K. Growth. $\lambda_{0x}, \lambda_{1x}, \lambda_{0x}^*, \lambda_{1x}^*$ are the risk parameters for the Forex factor. Robust standard errors are reported in the parentheses. *, ** and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

### Panel A: U.S.

<table>
<thead>
<tr>
<th>$\lambda_0 (\times 10^2)$</th>
<th>$\lambda_1$</th>
</tr>
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<tbody>
<tr>
<td>0.256</td>
<td>-0.001</td>
</tr>
<tr>
<td>(0.224)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>0.160</td>
<td>0.005</td>
</tr>
<tr>
<td>(0.042)**</td>
<td>(0.002)**</td>
</tr>
<tr>
<td>-0.158</td>
<td>0.645</td>
</tr>
<tr>
<td>(10.80)</td>
<td>(1.314)</td>
</tr>
<tr>
<td>0.189</td>
<td>0.087</td>
</tr>
<tr>
<td>(3.10)</td>
<td>(0.507)</td>
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</table>

<table>
<thead>
<tr>
<th>$\lambda_{0x} (\times 10^3)$</th>
<th>$\lambda_{1x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.060</td>
<td>-1.199</td>
</tr>
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<td>(19.60)</td>
<td>(0.008)**</td>
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### Panel B: U.K.

<table>
<thead>
<tr>
<th>$\lambda_0^* (\times 10^2)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0.222</td>
<td>-0.003</td>
</tr>
<tr>
<td>(0.074)**</td>
<td>(0.001)**</td>
</tr>
<tr>
<td>0.079</td>
<td>0.007</td>
</tr>
<tr>
<td>(0.091)</td>
<td>(0.007)</td>
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<tr>
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<td>5.246</td>
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<tr>
<td>(2.500)</td>
<td>(0.783)</td>
</tr>
<tr>
<td>2.673</td>
<td>-0.320</td>
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<tr>
<td>(2.200)</td>
<td>(0.899)</td>
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</table>

<table>
<thead>
<tr>
<th>$\lambda_{0x} (\times 10^3)$</th>
<th>$\lambda_{1x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.500</td>
<td>-1.065</td>
</tr>
<tr>
<td>(37.00)</td>
<td>(0.003)**</td>
</tr>
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</table>
Table 3: **Yield curve fitting**. The table reports bond pricing errors and standard deviations of measurement errors. The pricing errors are calculated as the RMSE between the model implied yields and the data. All quantities are annualized.

<table>
<thead>
<tr>
<th>Panel A: U.S.</th>
<th>Maturity (year)</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pricing errors (annualized bps)</td>
<td>13</td>
<td>7</td>
<td>7</td>
<td>12</td>
<td>11</td>
<td>7</td>
<td>7</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Measuring Errors (annualized bps)</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Panel B: U.K.</th>
<th>Maturity (year)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pricing errors (annualized bps)</td>
<td>11</td>
<td>5</td>
<td>9</td>
<td>7</td>
<td>4</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Measuring Errors (annualized bps)</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
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</table>

Table 4: **Loss of Fit**. The table reports the loss of fit measures. For each row, we fix one of the states listed in the first column at its unconditional mean, then we calculate the RMSE between the model implied value and the data. The row with state “Benchmark” represents the case when all states are activated.

<table>
<thead>
<tr>
<th>States</th>
<th>RMSE of return volatility</th>
<th>RMSE of implied variance($\times10^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>0.0131</td>
<td>0.5252</td>
</tr>
<tr>
<td>U.S. Level</td>
<td>0.0098</td>
<td>0.5072</td>
</tr>
<tr>
<td>U.S. Slope</td>
<td>0.0136</td>
<td>0.5086</td>
</tr>
<tr>
<td>U.S. Inflation</td>
<td>0.0231</td>
<td>0.5046</td>
</tr>
<tr>
<td>U.S. Growth</td>
<td>0.0111</td>
<td>0.6179</td>
</tr>
<tr>
<td>U.K. Level</td>
<td>0.0133</td>
<td>0.5082</td>
</tr>
<tr>
<td>U.K. Slope</td>
<td>0.0120</td>
<td>0.5094</td>
</tr>
<tr>
<td>U.K. Inflation</td>
<td>0.0277</td>
<td>0.5631</td>
</tr>
<tr>
<td>U.K. Growth</td>
<td>0.0131</td>
<td>0.4970</td>
</tr>
<tr>
<td>Latent</td>
<td>0.0466</td>
<td>0.5255</td>
</tr>
</tbody>
</table>
Table 5: **Return Predictability.** The table reports the results of the predictive regression: \( r_{t+1} = \alpha + \beta z_t + \epsilon_{t+1} \). The Newey-West t-statistics are reported in the parentheses. We suppress the sign of t-statistics. All predictors are from 1996M7-2015M12 except for the investor sentiment (IS), whose data is only available up to 2015M9. *, ** and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>Predictors ( z_t )</th>
<th>S&amp;P 500</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta(%) )</td>
<td>t-stat</td>
</tr>
<tr>
<td>LTR</td>
<td>0.14</td>
<td>0.60</td>
</tr>
<tr>
<td>INFL</td>
<td>-0.09</td>
<td>0.31</td>
</tr>
<tr>
<td>LTY</td>
<td>-0.25</td>
<td>0.95</td>
</tr>
<tr>
<td>SVAR</td>
<td>-0.69</td>
<td>1.77*</td>
</tr>
<tr>
<td>DE</td>
<td>0.04</td>
<td>0.08</td>
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<tr>
<td>DFY</td>
<td>-0.32</td>
<td>0.59</td>
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<tr>
<td>TBL</td>
<td>-0.16</td>
<td>0.53</td>
</tr>
<tr>
<td>DY</td>
<td>0.65</td>
<td>1.67*</td>
</tr>
<tr>
<td>EP</td>
<td>0.26</td>
<td>0.52</td>
</tr>
<tr>
<td>TMS</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>B/M</td>
<td>0.31</td>
<td>1.02</td>
</tr>
<tr>
<td>DP</td>
<td>0.58</td>
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<tr>
<td>NTIS</td>
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<td>1.26</td>
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<tr>
<td>IS</td>
<td>-0.59</td>
<td>1.83*</td>
</tr>
<tr>
<td>FX</td>
<td>-0.97</td>
<td>2.72***</td>
</tr>
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</table>
Table 6: Bivariate Predictive Regression. The table reports the results of the bivariate predictive regression: $r_{t+1} = \alpha + \beta F_{X_t} + \psi X_t + \epsilon_{t+1}$. The Newey-West t-statistics are reported in the parentheses. We suppress the sign of t-statistics. All predictors are from 1996M7-2015M12 except for the investor sentiment (IS), whose data is only available up to 2015M9. *, **, and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>Predictors(X)</th>
<th>S&amp;P 500</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta$ (%)</td>
<td>t-stat</td>
</tr>
<tr>
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<td>-0.97</td>
<td>2.61***</td>
</tr>
<tr>
<td>INFL</td>
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<td>2.75***</td>
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<td>2.92***</td>
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<tr>
<td>SVAR</td>
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<td>2.83***</td>
</tr>
<tr>
<td>DE</td>
<td>-0.97</td>
<td>2.65***</td>
</tr>
<tr>
<td>DFY</td>
<td>-0.94</td>
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</tr>
<tr>
<td>TBL</td>
<td>-0.96</td>
<td>2.66***</td>
</tr>
<tr>
<td>DY</td>
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<tr>
<td>EP</td>
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<td>2.57**</td>
</tr>
<tr>
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<td>2.83***</td>
</tr>
<tr>
<td>B/M</td>
<td>-0.96</td>
<td>2.65***</td>
</tr>
<tr>
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<td>2.39**</td>
</tr>
<tr>
<td>DUMMY</td>
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<td>2.50**</td>
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</table>
### Table 7: Long Horizon Return Predictability

The table reports the results of the long-horizon predictive regression $\frac{1}{h} \sum_{i=1}^{h} r_{t+i} = \alpha + \beta X_t + \epsilon_{t+i}$. The Newey-West t-statistics are reported in the parentheses. We suppress the sign of t-statistics. *, ** and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>Horizon(h)</th>
<th>S&amp;P 500</th>
<th>FTSE</th>
<th>S&amp;P 500 on Dollar factor</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$\beta$(%) (t-stat)</td>
<td>Adj $R^2$ (%)</td>
<td>$\beta$(%) (t-stat)</td>
</tr>
<tr>
<td>1</td>
<td>-0.97(2.72)***</td>
<td>4.16</td>
<td>-0.67(2.28)***</td>
</tr>
<tr>
<td>2</td>
<td>-0.87(2.09)**</td>
<td>6.46</td>
<td>-0.62(1.98)**</td>
</tr>
<tr>
<td>3</td>
<td>-0.86(2.23)**</td>
<td>9.51</td>
<td>-0.59(2.07)**</td>
</tr>
<tr>
<td>4</td>
<td>-0.76(2.20)**</td>
<td>9.39</td>
<td>-0.53(1.94)**</td>
</tr>
<tr>
<td>5</td>
<td>-0.69(2.24)**</td>
<td>9.21</td>
<td>-0.48(1.98)**</td>
</tr>
<tr>
<td>6</td>
<td>-0.64(2.29)*</td>
<td>9.12</td>
<td>-0.44(1.96)*</td>
</tr>
<tr>
<td>9</td>
<td>-0.46(2.16)**</td>
<td>6.45</td>
<td>-0.31(1.79)*</td>
</tr>
<tr>
<td>12</td>
<td>-0.36(1.94)*</td>
<td>5.16</td>
<td>-0.22(1.43)</td>
</tr>
<tr>
<td>24</td>
<td>-0.24(1.14)</td>
<td>3.93</td>
<td>-0.21(1.04)</td>
</tr>
<tr>
<td>36</td>
<td>-0.27(2.04)**</td>
<td>8.01</td>
<td>-0.29(2.23)**</td>
</tr>
</tbody>
</table>

### Table 8: Cross-section Predictability

The table reports the cross-sectional regression results. MKT, SMB, HML, RMW and CMA are the Fama-French five factors, FX is the Forex factor. Shanken (1992) corrected t-statistics are reported in the parentheses. We suppress the sign of t-statistics. *, ** and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>Constant</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
<th>RMW</th>
<th>CMA</th>
<th>FX</th>
<th>Adjusted $R^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.013</td>
<td>-0.003</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4.3</td>
</tr>
<tr>
<td>t-stat</td>
<td>(2.35)**</td>
<td>(0.99)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.013</td>
<td>-0.003</td>
<td></td>
<td></td>
<td></td>
<td>0.506</td>
<td>41.9</td>
</tr>
<tr>
<td>t-stat</td>
<td>(2.09)**</td>
<td>(0.93)</td>
<td></td>
<td></td>
<td></td>
<td>(1.57)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.017</td>
<td>-0.003</td>
<td>-0.004</td>
<td>-0.003</td>
<td></td>
<td></td>
<td>21.1</td>
</tr>
<tr>
<td>t-stat</td>
<td>(5.01)***</td>
<td>(0.75)</td>
<td>(0.73)</td>
<td>(0.95)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.013</td>
<td>-0.001</td>
<td>-0.002</td>
<td>-0.001</td>
<td></td>
<td></td>
<td>0.448</td>
</tr>
<tr>
<td>t-stat</td>
<td>(4.93)***</td>
<td>(0.32)</td>
<td>(0.31)</td>
<td>(0.36)</td>
<td></td>
<td></td>
<td>(1.57)</td>
</tr>
<tr>
<td>5</td>
<td>0.012</td>
<td>0.001</td>
<td>-0.002</td>
<td>-0.003</td>
<td>0.001</td>
<td>0.005</td>
<td>42.7</td>
</tr>
<tr>
<td>t-stat</td>
<td>(5.29)***</td>
<td>(0.03)</td>
<td>(0.37)</td>
<td>(0.58)</td>
<td>(0.22)</td>
<td>(1.07)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.012</td>
<td>0.001</td>
<td>-0.001</td>
<td>-0.002</td>
<td>0.001</td>
<td>0.004</td>
<td>0.251</td>
</tr>
<tr>
<td>t-stat</td>
<td>(5.02)***</td>
<td>(0.07)</td>
<td>(0.26)</td>
<td>(0.31)</td>
<td>(0.15)</td>
<td>(0.94)</td>
<td>(0.89)</td>
</tr>
</tbody>
</table>
Table 9: **Source of predictability.** The table reports the results of return decomposition. The Newey-West t-statistics are reported in the parentheses. *, **and *** indicate significance at the 10%, 5%, and 1% levels, respectively.

<table>
<thead>
<tr>
<th>zt</th>
<th>$\beta^E$ (%)</th>
<th>$\beta^{CF}$ (%)</th>
<th>$\beta^{DR}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No predictor</td>
<td>-0.11</td>
<td>-0.39</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>(-1.01)</td>
<td>(-2.40)**</td>
<td>(2.07)**</td>
</tr>
<tr>
<td>DY</td>
<td>-0.09</td>
<td>-0.40</td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>(-0.81)</td>
<td>(-2.64)**</td>
<td>(2.02)**</td>
</tr>
<tr>
<td>EP</td>
<td>-0.12</td>
<td>-0.87</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(-0.99)</td>
<td>(-1.81)*</td>
<td>(-0.09)</td>
</tr>
<tr>
<td>DE</td>
<td>-0.12</td>
<td>-0.87</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(-0.99)</td>
<td>(-1.81)*</td>
<td>(-0.09)</td>
</tr>
<tr>
<td>BM</td>
<td>-0.10</td>
<td>-0.42</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>(-0.96)</td>
<td>(-2.30)**</td>
<td>(1.74)*</td>
</tr>
<tr>
<td>TBL</td>
<td>-0.09</td>
<td>-0.40</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>(-0.80)</td>
<td>(-2.67)**</td>
<td>(2.01)**</td>
</tr>
<tr>
<td>DFY</td>
<td>-0.32</td>
<td>-0.64</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>(-1.90)*</td>
<td>(-2.65)**</td>
<td>(0.01)</td>
</tr>
<tr>
<td>LTY</td>
<td>-0.14</td>
<td>-0.47</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>(-1.27)</td>
<td>(-2.76)**</td>
<td>(1.55)</td>
</tr>
<tr>
<td>TMS</td>
<td>-0.03</td>
<td>-0.40</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>(-0.30)</td>
<td>(-2.38)**</td>
<td>(2.45)**</td>
</tr>
<tr>
<td>NTIS</td>
<td>-0.50</td>
<td>-0.25</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>(-2.84)**(-1.92)*</td>
<td>(0.68)</td>
<td></td>
</tr>
<tr>
<td>INFL</td>
<td>-0.11</td>
<td>-0.39</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>(-1.04)</td>
<td>(-2.42)**</td>
<td>(2.07)**</td>
</tr>
<tr>
<td>LTR</td>
<td>-0.11</td>
<td>-0.39</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>(-1.06)</td>
<td>(-2.42)**</td>
<td>(2.02)**</td>
</tr>
<tr>
<td>DFR</td>
<td>-0.11</td>
<td>-0.39</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>(-1.09)</td>
<td>(-2.41)**</td>
<td>(2.10)**</td>
</tr>
<tr>
<td>SVAR</td>
<td>-0.27</td>
<td>-0.32</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>(-1.80)*</td>
<td>(-1.99)**</td>
<td>(1.94)*</td>
</tr>
<tr>
<td>IS</td>
<td>-0.15</td>
<td>-0.39</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>(-1.27)</td>
<td>(-2.83)**</td>
<td>(1.64)</td>
</tr>
</tbody>
</table>
Fig. 1. **Fit of exchange rate return.** The figure plots the data and the model implied exchange rate return. Sample period: 1996M5-2016M2.
Fig. 2. **Variance decomposition of exchange rate return.** The figure plots the results of variance decomposition. The dash dot line (Model) represents the model-implied volatility of exchange rate return. The dash line (No Forex) represents the model-implied volatility when the Forex factor is fixed at its unconditional mean. The solid line (Data) is the volatility of the data.
Fig. 3. Predictability of characteristic portfolios. The figure reports the results of the predictive regression by using characteristic portfolios as the testing assets. The red line is the 5% threshold level.
Fig. 4. Conditional variance of log SDF and Forex factor. The figure plots the standardized conditional variance of model generated log SDF and the standardized Forex factor.
Fig. 5. **Forex versus Dollar factor.** The figure plots the scatter between the Dollar factor of Verdelhan (2016) and the Forex factor.
Fig. 6. **Performance of the particle filter**: The figure plots the filtered and true state from an arbitrarily selected set of simulated data.
Appendix A. Details of Predictors and industry portfolios

A.1. Predictors

We obtain data of most predictors from Amit Goyal’s website. The investor sentiment data is from Jeffery Wurgler’s website. The details of 14 predictors are listed below:

- Long term return (LTR): return on the long term government bond
- Inflation (INFL): calculated from CPI for all urban consumers, lagged for two months to wait for the CPI releases
- Long term yield (LTY): yield of long term government bond
- Stock variance (SVAR): constructed from the sum of squared daily returns of S&P 500
- Dividend-payout ratio (DE): difference between the log dividend and the log earnings
- Default yield spread (DFY): difference between the yields on BAA- and AAA-rated corporate bond
- Treasury bill rate (TBL): secondary market three-month Treasury bill rate
- Dividend yield (DY): difference between the log dividend and the log of lagged price
- Earnings price ratio (EP): difference between the log earnings and the log price
- Term spread (TMS): difference between long-term yield and Treasury bill rate
- Book-to-market ratio (BM): ratio of book value to market value for DJIA
- Dividend price ratio (DP): difference between the log dividend and log price
- Net equity expansion (NTIS): ratio of 12-month moving sums of net issues by NYSE listed stocks to end-of-year total market capitalization
- Investor sentiment (IS): constructed from the first principal component of five standardized sentiment proxies, where each of the proxies has first been orthogonalized with respect to a set of six macroeconomic indicators

A.2. Full details of 30 industry portfolios

- Food: Food Products
- Beer: Beer and Liquor
- Smoke: Tobacco Products
- Games: Recreation
- Books: Printing and Publishing
• Hshld: Consumer Goods
• Clths: Apparel
• Hlth: Healthcare, Medical Equipment, Pharmaceutical Products
• Chems: Chemicals
• Txtls: Textiles
• Cnstr: Construction and Construction Materials
• Steel: Steel Works Etc
• FabPr: Fabricated Products and Machinery
• ElcEq: Electrical Equipment
• Autos: Automobiles and Trucks
• Carry: Aircraft, ships, and railroad equipment
• Mines: Precious Metals, Non-Metallic, and Industrial Metal Mining
• Coal: Coal
• Oil: Petroleum and Natural Gas
• Util: Utilities
• Telcm: Communication
• Servs: Personal and Business Services
• BusEq: Business Equipment
• Paper: Business Supplies and Shipping Containers
• Trans: Transportation
• Whlsl: Wholesale
• Rtail: Retail
• Meals: Restaraunts, Hotels, Motels
• Fin: Banking, Insurance, Real Estate, Trading
• Other: Everything Else

Appendix B. Solve for $E_t^Q [\sigma_{t+1}^2]$

This appendix provides the derivation of risk-neutral one period ahead expected variance in affine model. First notice that:

$$E_t^Q [\sigma_{t+1}^2] = E_t [M_{t+1} \sigma_{t+1}^2] / E_t [M_{t+1}].$$  \hspace{1cm} (B.1)
We derive the expression for the nominator as follows, first we set \( a = \lambda_0 - \lambda_t^0 \), \( b = \lambda_1 - \lambda_t^1 \), then

\[
E_t[M_{t+1}\sigma^2_{t+1}] = E_t[\exp(-r_t - \frac{1}{2}\lambda_t^0\lambda_t - \lambda_t^0\epsilon_{t+1})(a + b'Z_{t+1})'(a + b'Z_{t+1})]
\]

\[
= \exp(-r_t - \frac{1}{2}\lambda_t^0\lambda_t)E_t[\exp(-\lambda_t^0\epsilon_{t+1})(a + b'\mu + \Phi Z_t + \Sigma\epsilon_{t+1})'(a + b'\mu + \Phi Z_t + \Sigma\epsilon_{t+1})]
\]

\[
= \exp(-r_t - \frac{1}{2}\lambda_t^0\lambda_t)E_t[\exp(-\lambda_t^0\epsilon_{t+1})(\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1} + 2(a + b'\mu + \Phi X_t)'b'\Sigma\epsilon_{t+1} + (a + b'\mu + \Phi Z_t))'(a + b'\mu + \Phi Z_t)].
\]

Thus we only need to calculate three terms involved in the last conditional expectation.

The first part is the quadratic term:

\[
E_t[\exp(-\lambda_t^0\epsilon_{t+1})\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}] = \int \exp(-\frac{1}{2}\epsilon_{t+1}\epsilon_{t+1} - \lambda_t^0\epsilon_{t+1})\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}d\epsilon_{t+1}
\]

\[
= \exp(-\lambda_t^0\lambda_t)\int \exp(-\frac{1}{2}(\epsilon_{t+1} + \lambda_t)'(\epsilon_{t+1} + \lambda_t))\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}d\epsilon_{t+1}.
\]

The integral actually computes \( E_t[\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}] \), whereas \( \epsilon_{t+1}|F_t \sim N(-\lambda_t, I) \), thus its expression is reduced to:

\[
E_t[\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}] = E_t[tr(b'\Sigma\epsilon_{t+1}\epsilon_{t+1}'\Sigma'b)] = tr(E_t[b'\Sigma\epsilon_{t+1}\epsilon_{t+1}'\Sigma'b]) = tr(b'\Sigma\Sigma'b + b'\Sigma\lambda_t\lambda_t^0b).
\]

(B.2)

So the first part will be:

\[
E_t[\exp(-\lambda_t^0\epsilon_{t+1})\epsilon_{t+1}'\Sigma'bb'\Sigma\epsilon_{t+1}] = \exp(-\frac{1}{2}\lambda_t^0\lambda_t)tr(b'\Sigma\Sigma'b + b'\Sigma\lambda_t\lambda_t^0b).
\]

Then the second part can be obtained similarly:

\[
E_t[\exp(-\lambda_t^0\epsilon_{t+1})(a + b'\mu + \Phi Z_t)'b'\Sigma\epsilon_{t+1}] = -\exp(-\frac{1}{2}\lambda_t^0\lambda_t)(a + b'\mu + \Phi Z_t)'b'\Sigma\lambda_t.
\]

The last part is constant term so it remains unchanged after taking expectations.

After using the fact that \( E_t[M_{t+1}] = \exp(-r_t) \), we could obtain the closed form expression.

\[
E_t[\sigma^2_{t+1}] = tr(b'\Sigma\Sigma'b + b'\Sigma\lambda_t\lambda_t^0b) - 2((a + b'(\mu + \Phi X_t))'b'\Sigma\lambda_t) + (a + b'(\mu + \Phi X_t))'(a + b'(\mu + \Phi X_t)).
\]

(B.3)
Appendix C. Proof of the Proposition

Denote \( \theta = \frac{1-\gamma}{1-\psi} \), \( G = \left( \frac{C}{(1-\gamma)(1-\psi)} \right)^{\frac{1}{1-\psi}} \), then the aggregator can be written as:

\[
f(C, J) = \beta \theta J(G - 1). \tag{C.1}
\]

The partial derivative is:

\[
f_J = (\theta - 1) \beta G - \beta \theta, \tag{C.2}
\]

\[
f_C = \frac{\beta G}{C}(1 - \gamma)J. \tag{C.3}
\]

Conjecture the value function has the form:

\[
J(W, y) = \exp(A_0 + A_1 y)\frac{W^{1-\gamma}}{1 - \gamma}. \tag{C.4}
\]

Using the envelope condition \( f_C = J_W \), we obtain

\[
\beta G = \frac{C_t}{W_t}, \tag{C.5}
\]

\[
C = J_W^{-\psi}((1 - \gamma)J)\frac{1-\psi}{1-\gamma} \beta \psi. \tag{C.6}
\]

Combine (C.4) and (C.6), we can express the value function as function of consumption:

\[
J(C, y) = \beta^{-\psi(1-\gamma)} \exp(\psi(A_0 + A_1 y))\frac{C^{1-\gamma}}{1 - \gamma}. \tag{C.7}
\]

Then (C.6) and (C.7) together will give the consumption-wealth ratio:

\[
\frac{C}{W} = \beta \psi \exp[(A_0 + A_1 y)\frac{1 - \psi}{1 - \gamma}]. \tag{C.8}
\]

We use the log-linear approximation as proposed in [Chacko and Viceira (2005)] and equation (C.5):

\[
\frac{C_t}{W_t} \approx g_1 - g_1 \log g_1 + g_1 \log(\beta G), \tag{C.9}
\]

where \( g_1 \) is the steady state consumption-wealth ratio.
On the other hand, the aggregator under such log-linearization is:

\[ f = \theta J(\beta G - \beta) \approx \theta J(g_1 - \beta - g_1 \log g_1 + g_1 \log \beta + g_1 \log G) = \theta J[g_1 \frac{1 - \psi}{1 - \gamma}(A_0 + A_1 y) + \xi], \]

(C.10)

where \( \xi = g_1 - g_1 \log g_1 + g_1 \psi \log \beta - \beta \).

Now we show how to find \( A_0 \) and \( A_1 \) in the conjectured solution (C.7). The HJB equation at optimal consumption is:

\[ f(C, J) + C \mu J_C + \frac{1}{2} yC^2 J_{CC} + (\sigma^2 - 2\kappa y) J_y + 2\sigma^2 y J_{yy} + 2\sigma cy FL' J_{Cy} = 0. \]

(C.11)

The solution (C.7) has the property:

\[ J_C = \frac{J(1 - \gamma)}{C}, \]

(C.12)
\[ J_{CC} = \frac{J(1 - \gamma)(-\gamma)}{C^2}, \]

(C.13)
\[ J_y = \psi A_1 J, \]

(C.14)
\[ J_{Cy} = \frac{\psi A_1 J(1 - \gamma)}{C}, \]

(C.15)
\[ J_{yy} = \psi^2 A_1^2 J. \]

(C.16)

Replace into (C.11), and use the log-linear approximation of \( f(C, J) \), we obtain:

\[ \theta J[g_1 \frac{1 - \psi}{1 - \gamma}(A_0 + A_1 y) + \xi] + J(1 - \gamma)\mu + \frac{1}{2} \gamma(\gamma - 1) J_y + 2\sigma^2 \psi^2 A_1^2 y J + 2\sigma \psi A_1 J(1 - \gamma) y FL' = 0. \]

(C.17)

Grouping the constant term gives \( A_0 \):

\[ A_0 = \frac{(\gamma - 1)(\theta \xi + (1 - \gamma)\mu)}{(1 - \psi)\theta g_1}, \]

(C.18)
\[ (C.19) \]

while \( A_1 \) is solved out from the quadratic equation:

\[ 2\sigma^2 \psi^2 A_1^2 + (\theta \psi_1 \frac{1 - \psi}{1 - \gamma} + 2\sigma \psi(1 - \gamma) FL') A_1 + \frac{1}{2} \gamma(\gamma - 1) = 0. \]

(C.20)
Then we solve for dynamics of state price density:

\[ \pi_t = \exp\left( \int_0^t f_J(C_s, J_s) ds \right) f_C(C_t, J_t). \]  

(C.21)

Applying Itô’s lemma

\[ \frac{d\pi_t}{\pi_t} = df_J + \frac{df_c}{f_c} + \frac{1}{2} df_J df_J + df_J \frac{df_c}{f_c}. \]  

(C.22)

For the two related terms on the right hand side, notice that from (C.2) and the log-linear approximation:

\[ f_J = (\theta - 1)\beta G - \beta \theta \approx (\theta - 1)(g_1 \frac{1 - \psi}{1 - \gamma} (A_0 + A_1 y) + \beta + \xi) - \beta \theta = \xi_1 - g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 y, \]  

(C.23)

\[ f_C = \beta \frac{G}{C}(1 - \gamma)J = \frac{(1 - \gamma)J}{W} = \beta^{\psi \gamma} \exp\left( \frac{1 - \gamma \psi}{1 - \gamma} (A_0 + A_1 y) \right) C^{-\gamma}. \]  

(C.24)

Then

\[ df_J = -g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 ((\sigma^2 - 2\kappa \psi \mu) dt + 2\sigma \sqrt{y} F dW_t), \]  

(C.25)

\[ \frac{df_c}{f_c} = -\gamma \frac{dC}{C} = -\gamma (\mu dt + \sqrt{y} L dW_t), \]  

(C.26)

\[ df_J df_J = (2g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 \sigma)^2 y dt, \]  

(C.27)

\[ \frac{df_J}{f_C} df_C = 2g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 \sigma \gamma y F L' dt. \]  

(C.28)

We thus obtain the state price density:

\[ \frac{d\pi}{\pi} = -r_t dt - (2g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 \sigma \sqrt{y} F + \gamma \sqrt{y} L) dW_t, \]  

(C.29)

where

\[ r_t = g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 (\sigma^2 - 2\kappa \psi \mu) - \gamma \mu + 2(g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 \sigma)^2 y + 2g_1 \frac{1 - \gamma \psi}{1 - \gamma} A_1 \sigma \gamma y F L'. \]  

(C.30)

Note that the short rate process is now quadratic in state, thus not consistent with equation (9). However, the specification of short rate process is not essential for the predictability results.
Thus the model-implied state price density matches the form in (31).

We study the equilibrium risk premium of consumption claim in this economy, applying Itô’s lemma on (C.8):

$$
\frac{d}{C} \frac{C}{W} = \frac{dC}{C} - \frac{dCdC}{2C^2} - \frac{dW}{W} + \frac{dWdW}{2W^2} = \frac{1 - \psi}{1 - \gamma} A_1 dy.
$$  \hspace{1cm} (C.31)

Suppose the wealth evolves according to the process:

$$
\frac{dW}{W} = \mu dt + \sigma H dW_t,
$$  \hspace{1cm} (C.32)

where $H$ is $1 \times m$ vector with $HH' = 1$. Equation (C.31) and (C.32) together imply the expressions for $\mu_t$ and $\sigma_t H$:

$$
\sigma_t H = \sqrt{\gamma_t L} - 4 \frac{1 - \psi}{1 - \gamma} A_1 \sigma \sqrt{\gamma_t F},
$$  \hspace{1cm} (C.33)

$$
\mu_t = \mu - \frac{1}{2} \gamma_t + \frac{1}{2} \sigma_t^2 - 2 \frac{1 - \psi}{1 - \gamma} A_1 (\sigma^2 - 2 \kappa y_t).
$$  \hspace{1cm} (C.34)

Under continuous time framework, the expected return for the consumption claim is:

$$
E_t \left( \frac{dW_t}{W_t} \right) + \frac{C_t}{W_t} dt = \tau dt - E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dW_t}{W_t} \right].
$$  \hspace{1cm} (C.35)

The risk premium is:

$$
E_t \left[ \frac{d\pi_t}{\pi_t} \frac{dW_t}{W_t} \right] = -\lambda_t KH' \sigma_t dt = -\lambda_t K (\sqrt{\gamma_t L'} - 4 \frac{1 - \psi}{1 - \gamma} A_1 \sigma \sqrt{\gamma_t F'})
$$

$$
= 4 \frac{1 - \psi}{1 - \gamma} A_1 \sigma \frac{\lambda_t^2 - \lambda_t \lambda_0}{\lambda_1} K F' - \frac{\lambda_t^2 - \lambda_t \lambda_0}{\lambda_1} K L'.
$$

Notice from (C.29) that the instantaneous variance of log state price density is linear function $y_t$ and therefore $\lambda_t^2$, therefore we find that indeed conditional variance of log SDF forecasts the stock risk premium.