Bargaining, Financing and Asset Prices: The Case of Real Estate

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Abstract

Real estate transactions are often established through financing. We study the effect of financing on property prices. We show that properties can transact at prices well above their collateral values. Therefore, the commonly used loan-to-value (LTV) ratio suffers a bias that understates credit risk. This bias is exacerbated when mortgages are originated with longer terms, at higher LTV ratios, or when sellers possess stronger bargaining power. Furthermore, this bias is larger under aggressive lending products, e.g. interest-only loans and mortgages allowing negative amortization. These findings call into questions underwriting and risk control practices in mortgages and other collateralized debts.

Keywords: asset prices, bargaining, mortgage financing, loan-to-value ratio, credit risk

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1. Introduction

Real estate transactions are often done through bargaining, and resulted transaction prices are usually financed through mortgage debt. Like all other debt, credit risk control is of paramount importance to mortgage lenders. In addition to assessing a borrower’s creditworthiness (e.g. FICO scores), a lender further controls credit risk by limiting the level of permissible financial leverage. Specifically, the loan amount must not exceed the collateral value of the property. It is a common practice for mortgage lenders to use transaction price as property’s collateral value and subsequently determine how much can be financed and at what interest rate. As a result, mortgage underwriting, both in the U. S. and around the world, often involves calculating the loan-to-value (LTV) ratio, which is the loan amount as a fraction of the transaction price of the property. Relative to many other asset classes, real estate is deemed more “collateralizable”, and ownerships of real estate are often accompanied by higher financial leverage. For example, conventional home loans in the U.S. can be originated with a LTV ratio of up to 80% without credit enhancements. With private mortgage insurance (PMI) or under the Federal Housing Administration (FHA) program or the Veteran’s Loan Guarantee Program, the LTV ratio can be well above 90%. Given the thin cushion between transaction price and the amount of debt, effective credit risk control critically relies on properties’ collateral values being accurately measured by transaction prices. But do transaction prices faithfully reflect collateral values?

In this study, we explore the price formation of a mortgage-financed property in a bilateral bargaining game. We show that a property can transact at a price that is well

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1 In Australia and New Zealand, the Loan-to-value ratio may be abbreviated as LVR.
above its collateral value, and the LTV ratio suffers a bias that understates credit risk. Our results indicate that the common practice of using the LTV ratio to gauge financial leverage is inaccurate. Consequently, credit risk control strategies based on LTV ratios, which are currently widely used in mortgage industries, are problematic. Our findings are critical for several reasons. First, the scale of mortgage market is vast, and ineffective control of credit risk of home loans has severe dire consequences, an example of which is the 2007 housing market meltdown and the slowdown of global economy many years after. Second, the bias of the LTV ratio that we identified here also exists in many other collateralized debt (e.g. commercial mortgages, car loans). Therefore, our results call into questions underwriting practices and risk control strategies of those debt instruments as well.

The deviation of transaction prices from collateral values is rooted in the fact that transaction prices are, at least in part, determined by financing. For example, low mortgage rates or attractive financing arrangements, *ceteris paribus*, reduces cost of ownership and, as a result, boost buyers’ willingness to acquire real estate. Buyers’ eagerness to buy, through price bargaining, translates to higher transaction prices. In other words, good financing terms create value, and such value is shared between the buyer and the seller and is reflected in transaction prices. It is worth noting that value created through financing is not intrinsic to the property and not a part of the property’s collateral value. Ebbs and flows of the economy drive mortgage rates up and down and cause various lending products to come and go. Therefore, prices formulated under favorable financing terms will overestimate collateral values when those financing

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arrangements cease to be available. Consequently, the LTV ratio contains a downward bias that makes lenders overoptimistic about the amount of credit risk they bear. Our simulation results suggest such a bias can be quite substantial. For example, a 30-year fixed rate mortgage (FRM) originated at an 80% LTV may in fact have its loan amount exceeding the collateral value of the property. That is, the loan is essentially “under water” at origination. Under higher leverages and more aggressive lending products, the degree to which a loan is “under water” at origination increases dramatically. For instances, at a LTV ratio of 95%, a 30-year FRM may disburse a loan amount that is over 126.13% of the collateral value of the property. In addition, a 30-year 80%-LTV interest-only loan may disburse a loan amount that is 108.97% of the collateral value of the property.

The bargaining-financing model we construct also allows us to explore the impact of bargaining and financing on the liquidity of real estate (i.e., time-on-market). To our knowledge, this is an area that has not been examined in the literature. Sellers of real estate conduct costly searches for buyers, and the benefit of search depends on the seller’s potential gain from trade, which is ultimately determined by transaction price. Therefore, if transaction price is jointly determined by bargaining and financing, so should liquidity. In our model, the seller optimizes her search time by taking into account the impact of both bargaining and financing. We find that stronger seller’s bargaining power and an increase in credit supply incentivize sellers to search more extensively and wait longer.

We contribute to existing literature in several ways. First, our model incorporates three critical elements of real estate transactions: search, bargaining and mortgage financing, into a unified framework to enable a holistic perspective on real estate
transaction and to gain in-depth understanding of how property prices and liquidity are determined. We show that transaction prices can be influenced by financing arrangements, which are unrelated to the fundamental value of a property. Because of that, transaction prices often fail to serve as an unbiased estimate of collateral value, and the LTV ratio understates credit risk. Our study is the first to identify and quantify such a bias contained in the widely-used LTV ratio for credit risk control. Additionally, we contribute to the literature by examining factors that influence the magnitude of such bias. Specifically, we show that the bias is larger when a mortgage is originated with a longer term, at a higher LTV ratio, or when the seller possesses stronger bargaining power relative to the buyer. Furthermore, this bias is larger under aggressive lending products, such as interest-only loans and mortgages that allow negative amortization. In addition, our analysis sheds light on an important but under-investigated question: How do bargaining and financing influence liquidity? We are able to derive a closed-form solution of optimal time-on-market that captures the impact of both bargaining and financing.

The second contribution of our study is that our theoretical framework adds new insights to the understanding of real estate market cycles. Our model can offer explanations why the pervasive use of aggressive lending products was a crucial contributing factor that fueled the dramatic escalation of U.S. housing prices during the early years of 2000s. It is worth noting that our explanation is different from ones offered by previous studies, which generally suggest that aggressive lending products boost housing demand by enabling risky borrowers who would previously have difficulty of obtaining a loan to become homeowners. The resulted additional demand bids up housing prices. We, on the other hand, suggest an alternative channel through which aggressive
lending products can incite a bubble. Having features that allow borrowers to delay repayments (e.g. interest-only (IO) loans and mortgages with negative amortization), aggressive lending products can be more attractive to some borrowers than traditional home loans. Through price bargaining, the value-added loan is split between the buyer and seller and translates into a higher transaction price. We show that aggressive lending products, even when only taken by high quality borrowers rather than risky borrowers and produce no additional demand, can inflate house prices.

This rest of the paper is organized as follows. Section 2 reviews related literature. In Section 3, we construct a model of real estate bargaining with mortgage financing and analyze the joint effect of bargaining and financing on property price formation and liquidity. In Section 4, we simulate our model under various scenarios to provide economic significance about the impact of bargaining and financing. Section 5 provides concluding remarks.

2. Literature Review

The effect of mortgage financing on property transaction prices has been examined by many empirical studies. It is well known that how a property is financed can have a substantial influence on its transaction price. Zerbst and Brueggeman (1977), Brueggeman and Zerbst (1979) and Colwell, Guntermann and Sirmans (1979) compare residential property transactions financed through Federal Housing Administration (FHA) loans or Veteran Affair (VA) loans, under which discount points are paid by sellers, to transactions financed through conventional mortgages. The studies find that sellers shift the cost of discount points to FHA buyers and VA buyers by adjusting prices upward.
Sirmans, Smith and Sirmans (1983) study house prices resulted from transactions in which assumable mortgages are used. They show that houses with assumable loans tend to sell at higher prices than the ones financed with conventional mortgages carrying higher interest rates. Rosen (1982) studies creative financing arrangements, such as teaser rates, assumable mortgages, and seller financing at below-market rates. He finds that the advantages of creative financing are capitalized in housing prices leading to a “creative financing premium”. More recent studies investigate the role of subprime mortgages in home prices and how they cause housing bubble conditions where the demand for subprime lending fueled lenders’ willingness to extend loans to more risky buyers, which in turn helped to further fuel the housing bubble and eventually led to the 2007-2009 housing crash due to borrowers’ defaults for various economic and behavioral reasons (e.g, Parlov and Wachter, 2011, Collins, Harrison and Seiler, 2015, and Seiler, 2015a, 2015b, 2017).

Our study relates to this stream of literature by examining the effect of financing from a theoretical perspective. First, our model provides a theoretical framework to help understand price formation of a mortgage-financed property. Although there is empirical evidence that advantageous financing arrangements are capitalized in transaction prices, the exact mechanism of such capitalization is less clear. Our model fills this gap by showing exactly how the capitalization of financing into prices is realized. Furthermore, our analysis goes beyond prices by looking at two other important issues. First, we study how transaction price can loop round and impact loan performance. For example, a transaction price substantially above the property’s collateral value may imply escalated likelihood of mortgage default. More importantly, the increased credit risk is not fully
reflected in the widely-used LTV ratio, and, as a result, lenders bear that risk but are uncompensated for. Second, we examine the effect of financing and bargaining on liquidity, which is largely overlooked in the literature. We fill this gap by deriving a closed-form solution of optimal time-on-market that captures the impact of both bargaining and financing. In addition, our findings can also explain recent empirical findings in the finance literature. For example, Min and Yang (2009) find that loss-given-default is positively related to the LTV ratio. We show that when the original LTV ratio is higher, the transaction price is more likely to be inflated. In a distressed housing market when house prices decline, high-LTV loans are of course more likely to default and produce a greater loss-given-default (because the collateral value is overstated due to financing). Adrian and Cowan (2004) focuses on default correlation (i.e. when defaults occur, how likely are they going to happen at the same time?). Our findings can explain why subprime mortgages have a high default correlation. In our model, high credit risk has nothing to do with idiosyncratic borrower characteristics (e.g. FICO scores and household income). Instead, risky loans are made due to a systematic flaw in mortgage underwriting (e.g. the LTV ratio understating credit risk). Therefore, when house prices start to fall, a lot of defaults are going to happen at the same time.

Our paper also contributes to the large body of literature on hedonic pricing models. Most hedonic pricing models do not explicitly take into account financing terms. Hansz and Hayunga (2016) show that only two studies control for financing terms in hedonic pricing models in the literature from 2000 to 2014. Our findings contribute to the thin literature on the effect of financing on home price formation, and highlight the importance of incorporating financing terms in hedonic pricing models.
Our study is also related to the literature on non-cooperative bargaining theory, which has been heavily influenced by the seminal work of Rubinstein (1982). Since Rubinstein (1982), the bargaining literature has been extended in various ways. In general, bargaining outcomes may be influenced by 1) outside options, 2) the rules of the bargaining process, and 3) information and market structures. Our model extends non-cooperative bargaining theory in a different direction. We show that there exists a feedback loop between bargaining and financing. On the one hand, the outcome of bargaining (e.g. transaction price) is affected by available financing arrangements, and more attractive financings lead to higher prices. On the other hand, transaction price determines how much can be financed (e.g. the loan amount) and the level of credit risk undertaken by the lender. Such an intertwined relationship between financing and bargaining has not been analyzed before, and our study attempts to fill this gap.

3. The Model

3.1 The Setup

The real estate market distinguishes itself from the financial market in its high degree of illiquidity. Unlike traders in the financial market who can readily buy or sell a security at its equilibrium price, the seller in the real estate market must spend time to find a desirable buyer. During the search process, the seller receives offers over time from a stream of potential buyers whose offering prices and timing of arrival are

\[\text{See Binmore (1985).}\]
\[\text{See Binmore, Rubinstein and Wolinsky (1986), Baliga and Serrano (1995) and Krishna and Serrano (1996).}\]
stochastic in nature. Suppose that \( t_j \) is the waiting time between the arrival of the \((j-1)\)th and \( j\)th buyers, then the random arrival time of the \( N\)th buyer is \( T_N = \sum_{j=1}^{N} t_j \). At time \( T_N \), the seller accumulates \( N \) offers, and note that \( T_N \) is a random variable in our model. In subsection 3.2, we will discuss how the seller will optimally choose \( N^* \).

Suppose that seller’s second-best use of the property is \( R \);\(^6\) and for buyer \( j \) (\( j = 1,2,3,...N \)), we denote \( V_j \) as his private value of the property. If buyer \( j \) pays all cash for the property with price \( P_j \), his gain and the seller’s gain from this transaction are respectively

\[
\Pi_j^B = V_j - P_j
\]  

and

\[
\Pi_j^S = P_j - R
\]

Therefore, the total gain from trade, \( \Pi_j \), is the summation of \( \Pi_j^B \) and \( \Pi_j^S \).

\[
\Pi_j = \Pi_j^B + \Pi_j^S = V_j - R
\]

Among the \( N \) buyers, some may find other appropriate properties and, as a result, exit the bidding. We denote \( \zeta \) as the percentage of bidders who are still interested in the property, thus \((1-\zeta)\) is the percentage of exiting offers. Other things being equal, a higher \( \zeta \) implies fewer exiting offers and a smaller supply of similar properties, hence a tighter housing market. Therefore, \( N(1-\zeta) \) is the number of available offers at waiting

\(^6\) Though out the paper, we use “she” when referring to the seller and “he” when referring to the buyer.
time on market $T_N$, where $T_N = \sum_{j=1}^{N} t_j$. We denote $V^\text{max} = \max\{V_1, V_2, \ldots, V_N\}$, and the total gain reaches the highest when the buyer has the highest valuation of the property among available buyers, i.e.,

$$\Pi^\text{max} = V^\text{max} - R \quad (4).$$

There will be no trade if there is no gain from the trade, i.e., $\Pi^\text{max} = V^\text{max} - R < 0$ or $V^\text{max} < R$. The necessary condition for the trade to occur is the existence of non-negative gain from the trade, $V^\text{max} \geq R$. In addition, suppose that the offer price for the buyer who has the highest valuation of the property is $P$. Since both the buyer’s gain and the seller’s gain should be non-negative, we have $V^\text{max} - P \geq 0$ and $P - R \geq 0$. As a result, $P$ must satisfy,

$$V^\text{max} \geq P \geq R \quad (5).$$

Therefore, with cash transaction the price is bounded by seller’s and buyer’s valuations of the property. In particular, it is above the seller’s valuation but below the buyer’s valuation.

In reality, cash purchases are not very common, and mortgages are often used to finance acquisitions of real estate. To introduce financing in the model, suppose that the buyer makes an equity investment, $E$, and finances the rest of the purchase price by borrowing $P - E$ at an interest rate $i$ under a $T$-period conventional fixed-rate mortgage (FRM). $^8$

With the $T$-period FRM, the periodic mortgage payment, $\text{PMT}$, must satisfy,

\(^7\) In the remainder of the paper, when referring to the buyer we mean the buyer who has the highest valuation of the property and is expected to outbid the others.

\(^8\) For the conventional FRM, we mean a fully-amortizing fixed-rate mortgage with a constant periodic payment. For simplicity, we consider a conventional FRM first. Financing through other types of mortgages will be discussed later in the paper.
\[ P - E = \sum_{r=1}^{T} \frac{PMT}{(1 + i)^r} \] 

(6).

Hence,

\[ PMT = \frac{(P - E)}{\sum_{r=1}^{T} \frac{1}{(1 + i)^r}} \] 

(7).

Suppose that the buyer has a discount rate of \( r \) for his future cash flows, the present value of his mortgage payments can be calculated as follows,

\[ PV_B = \sum_{r=1}^{T} \frac{PMT}{(1 + r)^r} \] .

(8).

Substituting Equation (7) into Equation (8) yields,

\[ PV_B = \frac{\sum_{r=1}^{T} \frac{1}{(1 + r)^r}}{\sum_{r=1}^{T} \frac{1}{(1 + i)^r}} \] 

(9).

To simplify notation, we adopt two definitions: \( \Omega_r \equiv \sum_{r=1}^{T} \frac{1}{(1 + r)^r} \) and \( \Omega_i \equiv \sum_{r=1}^{T} \frac{1}{(1 + i)^r} \).

Therefore, Equation (9) can be rewritten as

\[ PV_B = (P - E) \frac{\Omega_r}{\Omega_i} \] 

(9)’.

With financing, the seller’s gain from trade remains to be \( \Pi^S = P - R \). However, the buyer’s gain from trade becomes

\[ \Pi^{B,\text{max}} = V^{\text{max}} - E - (P - E) \frac{\Omega_r}{\Omega_i} \] .

(10).

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9 We assume the source of the buyer’s financing is a third-party financial institution, not the seller.
Equation (10) indicates that the buyer’s gain is equal to his valuation of the property, 
\[ V_{\text{max}} \], less two costs: 1) equity investment, \( E \), and 2) the present value of mortgage repayment to the buyer.

**Theorem 1.** With mortgage financing, the total gain from trade becomes,

\[
\Pi_{\text{max}} = V_{\text{max}} - R + \left(1 - \frac{\Omega}{\Omega_i}\right)(P - E) \tag{11}
\]

Financing creates new ground for trading. In contrast to conventional wisdom, a buyer does not have to value a property more than its seller for a mutually beneficial trade to occur, and transaction price is not bounded and it can be above the buyer’s valuation.

**Proof:**

Comparing Equation (11) to Equation (4), we observe that the introduction of mortgage financing into the model transforms the total gain from trade from \( V_{\text{max}} - R \) to \( V_{\text{max}} - R + \left(1 - \frac{\Omega}{\Omega_i}\right)(P - E) \). This transformation suggests: First, mortgage financing alters the necessary condition under which a mutually beneficial trade can occur. With cash purchase, there will be no transaction if \( V_{\text{max}} < R \). However, that is no longer the case when mortgage financing is used. Even with \( V_{\text{max}} < R \), we can still have \( \Pi_{\text{max}} > 0 \) if \( \left(1 - \frac{\Omega}{\Omega_i}\right)(P - E) > R - V_{\text{max}} \). In other words, favorable mortgage financing makes the impossibility of trading under cash transaction possible. Second, with financing,
transaction price, $P$, is no longer bounded by the seller’s and the buyer’s valuations of the property. To see this, we rewrite Equation (10) as

$$\Pi^{B,\max} = V^{\max} - P + (1 - \frac{\Omega_i}{\Omega})(P - E).$$

Equation (12) implies that when $\left(1 - \frac{\Omega_i}{\Omega}\right)(P - E)$ is strictly greater than zero, the transaction price $P$ can be higher than the buyer’s value of the property, $V^{\max}$, because it is still possible for the buyer to have a positive gain from the transaction as long as Equation (12) is strictly positive.

End of the proof

Therefore, Theorem 1 indicates when mortgage financing is used, transaction prices reflect not only property values but also how the financing is structured.

3.2 Seller’s Optimal Waiting Time on Market and Price Formation

A defining feature of real estate transaction process is the sequential but random arrival of potential buyers. Both offer prices and the time of arrival are random. The first question faced by the seller is when to stop the search and start negotiating with available buyers. On the one hand, the seller should wait for as many buyers as possible to have the chance of encountering the highest offer price. However, waiting is costly, and beyond a certain point, the marginal benefit of waiting for another buyer may no longer justify the cost of doing so. In other words, the seller first needs to determine when is the optimal time to stop the search and start negotiation.
Similar to Arnold (1999), we assume that buyer $j$’s valuation, $V_j$, is distributed over $[V_j, \overline{V}]$ with mean $\mu$ and variance $\sigma^2$ ($j = 1, 2, \ldots, N$), where $V_j$ and $\overline{V}$ are respectively the lower and upper bounds. Suppose that $f(V_j)$ and $F(V_j)$ are probability density function and cumulative distribution function, respectively. Given that $V_{\text{max}}$ is the highest valuation among $N\zeta$ available buyers, the density function of $V_{\text{max}}$ can be expressed as follows,

$$g_{V_{\text{max}}}(V) = N\zeta F(V)^{N\zeta - 1} f(V)$$

(13)\textsuperscript{10}.

We thus have

$$E[V_{\text{max}}] = \int_{V}^{\overline{V}} V N\zeta F(V)^{N\zeta - 1} f(V) dV$$

(14).

We next evaluate the seller’s marginal benefit of waiting for the $N$th buyer’s arrival. We model the bargaining between the seller and the buyer by using the non-cooperative bargaining model of Rubinstein (1982). If the total gain from trade, $\Pi_{\text{max}}$ in Equation (11), is greater than zero, the seller and the buyer will engage in a bargaining game described in Rubinstein (1982) to divide the gain. The seller first makes an offer by proposing a split $(x, \Pi_{\text{max}} - x)$. If the buyer accepted the proposal, the game ends and a deal is reached. The payoffs received by the seller and the buyer are $x$ and $\Pi_{\text{max}} - x$ respectively. If the proposal is rejected, the buyer makes a counter-offer by proposing a new split $(x', \Pi_{\text{max}} - x')$. If the seller accepts it, the game ends, and the payoff to the seller and the buyer are now equal to $\theta_{S}^{Bgn} x'$ and $\theta_{B}^{Bgn} (\Pi_{\text{max}} - x')$, where $\theta_{S}^{Bgn} < 1$ and

\textsuperscript{10} See Ross (2002, p. 275).
\( \theta_B^{Bgn} < 1 \) respectively represent the seller’s and the buyer’s bargaining power. There are many factors affecting \( \theta_S^{Bgn} \) and \( \theta_B^{Bgn} \) including seller’s and buyer’s search cost and their anxiety of waiting. Alternatively, if the seller rejects the offer, she makes a counter offer. Thereafter, the seller and the buyer take turns to make proposals.

We follow the technique developed by Shaked and Sutton (1984) to determine the perfect equilibrium of this bargaining game. We define \( M \) as the supremum of the payoffs that the seller can obtain in any perfect equilibrium of the game. Suppose the buyer rejects the seller’s first offer, the seller’s payoff becomes \( \theta_S^{Bgn} M \). Now consider a proposal made by the buyer after rejection. Any split proposed by the buyer which gives the seller more than \( \theta_S^{Bgn} M \) will be accepted by the seller, so there is no perfect equilibrium in which the seller receives more than \( \theta_S^{Bgn} M \). It follows that the buyer will get at least \( \Pi^{\max} - \theta_S^{Bgn} M \) in any perfect equilibrium of the subgame beginning from that point. In fact, \( \Pi^{\max} - \theta_S^{Bgn} M \) is the infimum of the payoff received by the buyer in this subgame.

Now consider a proposal made by the seller in the first offer. Any split proposed by the seller which gives the buyer anything less than \( \theta_B^{Bgn} (\Pi^{\max} - \theta_S^{Bgn} M) \) will not be accepted by the buyer. Hence the seller will obtain at most \( \Pi^{\max} - \theta_B^{Bgn} (\Pi^{\max} - \theta_S^{Bgn} M) \). In fact, as before, this is the supremum of what the seller will receive in the first offer.

But the game of the seller’s second offer is identical to the game of the seller’s first offer, apart from shrinkage of all payoffs due to \( \theta_B^{Bgn} \) and \( \theta_S^{Bgn} \). Hence, it follows that the supremum of the seller’s payoff must equal \( M \). Therefore, \( M \) should satisfy,

\[
M = \Pi^{\max} - \theta_S^{Bgn} (\Pi^{\max} - \theta_S^{Bgn} M)
\]  

(15).
Solving for \( M \), we have 
\[
M = \frac{1 - \theta_B^{\text{Bgn}}}{1 - \theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} \Pi^\text{max}.
\]
Hence, the split of the gain between the seller and the buyers is as follows,

a) The seller’s gain from trade is:

\[
\Pi_S = M = \frac{1 - \theta_B^{\text{Bgn}}}{1 - \theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} \Pi^\text{max} \tag{16}.
\]

b) The buyer’s gain, which is simply the difference between the total gain and the seller’s gain, is,

\[
\Pi_B = \Pi^\text{max} - \Pi_S = \frac{\theta_B^{\text{Bgn}} (1 - \theta_S^{\text{Bgn}})}{1 - \theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} \Pi^\text{max} \tag{17}.
\]

Equations (16) and (17) indicate the seller’s and the buyer’s gain from trade depend on two things: 1) the size of total gain from trade, \( \Pi^\text{max} \); 2) the seller’s and the buyer’s bargaining power (\( \theta_S^{\text{Bgn}}, \theta_B^{\text{Bgn}} \)). In particular, both the seller and the buyer receive a greater payoff if the total gain from trade is greater. In addition, seller’s (buyer’s) gain decreases with buyer’s (seller’s) bargaining power and increases with her (his) own bargaining power. Intuitively, the seller (the buyer) receives a larger share if she (he) is more patient or if the buyer (the seller) is less patient.

With Equation (14), we can rewrite Equation (11) as follows,

\[
\Pi^\text{max} = E[V^\text{max}] - R + (1 - \frac{\Omega}{\Omega_i})(P - E) \tag{11}'.
\]

From Equations (11)’ and (16), the seller’s expected gain from trade is,

\[
\Pi_S = \frac{1 - \theta_B^{\text{Bgn}}}{1 - \theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} \left\{ E[V^\text{max}] - R + (1 - \frac{\Omega}{\Omega_i})(P - E) \right\} \tag{18}.
\]
Note that seller’s gain is also equal to \( P - R \). Therefore, the transaction price, \( P \), must satisfy,

\[
P - R = \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left\{ E[V_{\text{max}}] - R + \left(1 - \frac{\Omega_{r}}{\Omega_{i}}\right)(P - E) \right\}
\]  

(19).

Solving for \( P \) from Equation (19), we can obtain a closed-form solution of the transaction price when bargaining and mortgage financing are considered simultaneously:

\[
P = \frac{R - \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left[1 - \frac{\Omega_{r}}{\Omega_{i}}\right] E + \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left(E[V_{\text{max}}] - R\right) \right\}}{1 - \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left[1 - \frac{\Omega_{r}}{\Omega_{i}}\right]}
\]  

(20).

Therefore, the marginal benefit of waiting for the \( N \)th buyer’s arrival can be estimated by,

\[
\frac{\partial P}{\partial N} = \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left(\frac{\partial E[V_{\text{max}}]}{\partial N}\right) - \left(1 - \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left[1 - \frac{\Omega_{r}}{\Omega_{i}}\right]\right) \right) \right) / \partial N
\]  

(21).

Substituting Equation (14) into Equation (21), we have,

\[
\frac{\partial E[P]}{\partial N} = \frac{1 - \theta_{B}^{\text{Bgn}}}{1 - \theta_{S}^{\text{Bgn}} \theta_{B}^{\text{Bgn}}} \left(\frac{\partial}{\partial \text{V}} \int_{V_{\text{min}}}^{V_{\text{max}}} \frac{\partial f(V) N_{\text{max}}}{N_{\text{max}} - 1} dV\right) \right) \right) / \partial N
\]  

(22).
As we know, the total waiting time for the \( N \)th buyer’s arrival is \( T_N = \sum_{j=1}^{N} t_j \). For the distribution of \( t_j \), Bond et al. (2007) examine the U.K. data and find that the buyer’s stochastic arrival \( t_j \) follows the Poisson process with rate \( \lambda \). Therefore,

\[
E[T_N] = \sum_{j=1}^{N} E[t_j] = \frac{N}{\lambda} \tag{23}
\]

Following Haurin (1988), suppose that the seller’s holding cost of waiting for the \( N \)th buyer’s arrival is \( g(T_N) = cT_N \), where \( c \) is the holding cost per unit of time. The expected holding cost of waiting for the \( N \)th buyer’s arrival can thus be estimated as follows,

\[
E[g(T_N)] = \frac{cN}{\lambda} \tag{24}
\]

Since the optimal \( N^* \) must meet the condition of “marginal benefit of waiting for the \( N \)th buyer’s arrival = marginal cost of waiting for the \( N \)th buyer’s arrival”, \( N^* \) should satisfy the following condition,

\[
\frac{1 - \theta_{\text{Bgn}}}{1 - \theta_{\text{Bgn}} \theta_{\text{Bgn}}} \left( \frac{\partial}{\partial N} \int \frac{\bar{V} N \zeta F(V)^{N \zeta - 1} f(V) dV}{\zeta N + V} \right) = \frac{c}{\lambda} \tag{25}
\]

Read (1988), and Lin and Vandell (2007) assume that buyer’s offer price is uniformly distributed. If this is the case, we can show that,

\[
\int_{\bar{V}}^{\bar{V}} \frac{\bar{V} N \zeta F(V)^{N \zeta - 1} f(V) dV}{\zeta N + V} = \frac{\zeta N + V}{\zeta N + 1} \tag{26}^{11}
\]

Therefore,

\[11\text{ See the proof in Appendix I.}\]
\[
\frac{\partial}{\partial N} \int_{V} VN\zeta F(V)^{N-1} f(V) dV = \frac{\zeta (V - V)}{(\zeta N + 1)^2}
\] (27)

Since \( \sigma^2 = \frac{(V - V)^2}{12} \), we have \( (V - V) = 2\sqrt{3}\sigma \). Therefore, we can rewrite Equation (27) as follows,

\[
\frac{\partial}{\partial N} \int_{V} VN\zeta F(V)^{N-1} f(V) dV = \frac{2\sqrt{3}\zeta \sigma}{(\zeta N + 1)^2}
\] (28)

From Equations (25) and (28), we can solve the optimal \( N^* \) and the seller’s expected optimal waiting time on market in Theorem 2.

**Theorem 2.** The optimal number of potential buyers \( (N^*) \) the seller should wait can be expressed as follows,

\[
N^* = \frac{1}{\zeta} \left[ 2\sqrt{3} \left( \frac{1 - \theta_B^{\text{Bgn}}}{1 - \theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} \right) \lambda \zeta \sigma \right]^{-1}
\] (29)

Where \( \zeta \) is the percentage of buyers who are still available, \( \lambda \) is the buyers’ arrival rate, \( c \) is the holding cost per unit of time, \( \sigma \) is the volatility of offer prices, and \( \theta_S^{\text{Bgn}} \) and \( \theta_B^{\text{Bgn}} \) respectively represent the seller’s and the buyer’s bargaining power.

In addition, From Equations (23) and (29), the seller’s expected optimal waiting time on market is,
Theorem 2 suggests that the expected optimal waiting time on market is affected by many factors such as seller’s holding cost ($c$), market conditions ($\lambda$), the tightness of the market ($\zeta$), and the dispersion of buyers’ offer prices. More importantly, $E[T_{N^*}]$ is also determined by both bargaining and financing. First, $E[T_{N^*}]$ is increasing in seller’s bargaining power and decreasing in the buyer’s bargaining power. This is consistent with our common intuition. Stronger seller bargaining power allows her to get a bigger share of the pie. As a result, she is incentivized to search longer and more extensively to make the pie larger. Second, Equation (30) also reveals that $E[T_{N^*}]$ is strictly decreasing in mortgage rate, $i$. This suggests that increase in credit supply (i.e., lower mortgage rates), ceteris paribus, make sellers want to search more extensively and wait longer for potential buyers.

Under the seller’s optimal $N^*$, we can express the resulted transaction price as follows,

$$P = \frac{R - \left(1 - \frac{\Omega_r}{\Omega_i}\right)E + \frac{1 - \theta^B_\text{Bgn}}{1 - \theta^S_\text{Bgn} \theta^B_\text{Bgn}} \left( E[V^{\max}[N^*] - R \right) }{1 - \left(1 - \frac{\Omega_r}{\Omega_i}\right) \left(1 - \frac{1 - \theta^B_\text{Bgn}}{1 - \theta^S_\text{Bgn} \theta^B_\text{Bgn}} \left[1 - \frac{\Omega_r}{\Omega_i}\right] \right)}$$  (31).

Four observations can be made on Equation (31). First, consistent with standard bargaining theory, we show that $P$ is strictly increasing in the bargaining power of the
seller (i.e., \( \partial P / \partial \theta^\text{Bgn}_S > 0 \)) and strictly decreasing in the bargaining power of the buyer (i.e., \( \partial P / \partial \theta^\text{Bgn}_B < 0 \)). Second, the transaction price increases with the underlying value of the property (i.e., \( \partial P / \partial R > 0 \)). Third, the transaction price increases with the buyer’s discount rate for his future mortgage payments (i.e., \( \partial P / \partial r > 0 \)). Finally, holding other things being equal, transaction price can be far away from the underlying value of the property \( E[V^\text{max}] \) or \( R \) when buyer’s discount rate \( r \) is relatively bigger and his down-payment \( E \) is small.

3.3 The Bias of Price-Based Loan-To-Value (LTV) Ratio

With the transaction price under the seller’s optimal \( N^* \) in Equation (31), we next derive the bias of price-based LTV ratio. For mathematical simplicity, suppose that both the seller and the buyer agree on the underlying value of the property, i.e., \( E[V^\text{max}]|N^*] = R \). In other words, the property is worth \( E[V^\text{max}]|N^*] \) or \( R \), and it should be the property’s collateral value. Therefore, Equation (31) becomes,

\[
P = \frac{R - \frac{1 - \theta^\text{Bgn}_B}{1 - \theta^\text{Bgn}_B \theta^\text{Bgn}_B} \left[ \frac{1 - \frac{\Omega_r}{\Omega_i}}{1 - \frac{\Omega_r}{\Omega_i}} \right] E}{1 - \frac{1 - \theta^\text{Bgn}_B}{1 - \theta^\text{Bgn}_B \theta^\text{Bgn}_B} \left[ \frac{1 - \frac{\Omega_r}{\Omega_i}}{1 - \frac{\Omega_r}{\Omega_i}} \right] E}
\]

(32).

Because \( E[V^\text{max}]|N^*] \) or \( R \) is unobservable in the market, only transaction price is revealed after bargaining between the seller and buyer. From Equation (32), in theory we

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12 Our main results are unchanged if seller and buyer do not agree on the underlying value of the properties: either buyer’s valuation > seller’s valuation or buyer’s valuation < seller’s valuation.
can impute the underlying value of the property \( R \) or \( E[V^\text{max} \mid N^+] \) from the observable transaction price \( P \) as follows,

\[
R = P - \frac{1 - \theta_{\text{Bgn}}^B}{1 - \theta_{\text{Bgn}}^S \theta_{\text{Bgn}}^E} (1 - \frac{\Omega}{\Omega_i}) (P - E)
\]  

(33).

Because \( R \) (i.e., the underlying value of the property) is unobservable, transaction price is commonly used as a proxy of collateral value to control for borrower’s credit risk.

Given the fact that the second term in right-hand side of Equation (33) is always positive\(^{13}\), the finding in Equation (33) suggests that transaction price is likely to overestimate collateral value \( (R) \) and result in a LTV ratio that is too low. As a result, high default risks could hide behind a seemingly reasonable LTV ratio. We next examine how the price-based LTV ratio underestimates true financial leverage and hence underestimates the borrower’s default risk.

Since the underlying value of the property is \( R \) or \( E[V^\text{max} \mid N^+] \), the true LTV ratio should be calculated as,

\[
\text{LTV}^{\text{true}} = \frac{P - E}{R}
\]  

(34).

However, the underlying value of the property cannot be observed in the market, in practice the transaction price is used instead to calculate the LTV ratio, i.e.,

\[
\text{LTV}^{\text{price-based}} = \frac{P - E}{P}
\]  

(35).

The difference between Equations (34) and (35) is the bias of the LTV ratio caused by this common practice, namely,

\(^{13}\) When \( r \) is strictly less than \( i \), i.e., \( \Omega_r / \Omega_i > 1 \), the buyer will not choose mortgage financing because he will be worse off by taking it.
When mortgage financing is used, the buyer is willing to pay more for the property. As a result, \( P > R \). In other words, \( \text{LTV}^{\text{bias}} < 0 \), and the \( \text{LTV}^{\text{price-based}} \) is likely to underestimate the true loan-to-value ratio and thus fails to capture the credit risk of the loan. From Equation (33), we can rewrite Equation (36) as follows,

\[
\text{LTV}^{\text{bias}} = (P - E) \left( \frac{1}{P} - \frac{1}{\frac{1-\theta^\beta_{\text{bgn}}}{1-\theta^\beta_{\text{bgn}} \theta^\beta_{\text{bgn}}} \left(1-\frac{\Omega}{\Omega_{i}}\right)(P - E)} \right)
\]

Note that \( P - E \) is the loan amount, which can be rewritten as \( \text{LTV}^{\text{price-based}} P \). Substituting it into Equation (37), we can obtain a closed-form solution of the LTV bias in Theorem 3.

**Theorem 3.** The LTV bias of FRM mortgages can be expressed as follows,

\[
\text{LTV}^{\text{bias}} = \text{LTV}^{\text{price-based}} \left( 1 - \frac{\frac{1-\theta^\beta_{\text{bgn}}}{1-\theta^\beta_{\text{bgn}} \theta^\beta_{\text{bgn}}} \left(1-\frac{\Omega}{\Omega_{i}}\right)\text{LTV}^{\text{price-based}}}{1-\frac{1-\theta^\beta_{\text{bgn}}}{1-\theta^\beta_{\text{bgn}} \theta^\beta_{\text{bgn}}} \left(1-\frac{\Omega}{\Omega_{i}}\right)\text{LTV}^{\text{price-based}}} \right)
\]

### 3.4 Alternative Lending Products

Now we consider a more general type of mortgage in which the buyer borrows \( (P - E) \) at an interest rate \( i \) with varying mortgage payments. Suppose that the term of

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14 This result is consistent with the findings of many empirical studies that properties with mortgage financing always sell higher prices than those with cash transactions. For example, Lusht and Hansz (1994) find a 16.5% discount for cash-only transactions in Lehigh County, Pennsylvania. More recently, Aroul and Hansz (2011) and Jauregui et al (2017) find 13.5% and 9% discount for cash transactions in Clovis, California and Franklin County, Ohio, respectively.
the mortgage is \( T \), and the buyer’s mortgage payments from period 1 to \( T \) are denoted as \( PMT_1, PMT_2, \ldots, PMT_t, \ldots PMT_{T-1}, PMT_T \), then \( PMT_t (t = 1, 2, \ldots, T) \) must satisfy,

\[
P - E = \sum_{t=1}^{T} \frac{PMT_t}{(1 + i)^t}
\]

(39).

If the buyer has an alternative mortgage with everything else being equal but with different payments in the periods \( j \) and \( k \) as follows,

\[
PMT_t^{alt} = PMT_t \text{ for all } t \in \{1, 2, \ldots, j - 1, j + 1, \ldots, k - 1, k + 1, \ldots, T\}, \]

\[
PMT_j^{alt} \neq PMT_j \text{ and } PMT_k^{alt} \neq PMT_k \quad (k > j)
\]

(40).

Suppose that the alternative mortgage pays less at period \( j \) with delaying payments, i.e., \( PMT_j^{alt} < PMT_j \). Since the loan amount for the two mortgages is the same, we thus have,

\[
\sum_{t=1}^{T} \frac{PMT_t^{alt}}{(1 + i)^t} = \sum_{t=1}^{T} \frac{PMT_t}{(1 + i)^t}
\]

(41).

We can readily obtain,

\[
PMT_t^{alt} = PMT_t \text{ for all } t \in \{1, 2, \ldots, j - 1, j + 1, \ldots, k - 1, k + 1, \ldots, T\}, \]

\[
PMT_j^{alt} < PMT_j \text{ and } PMT_k^{alt} > PMT_k \quad (k > j)
\]

(42).

**Theorem 4.** Both the buyer’s gain and the total gain from trade are higher under the mortgages with delaying payments. As a result, the more aggressive lending products such as interest-only loans and mortgages that allow negative amortization (i.e., perfect examples of the mortgages with delaying payments) will lead to higher transaction prices.
Proof:

Both mortgages have the same loan amount of \((P - E)\), we thus have

\[
\sum_{t=1}^{T} \frac{PMT_t}{(1+i)^t} = P - E = \sum_{t=1}^{T} \frac{PMT_{t}^{alt}}{(1+i)^t}
\]  

(43)

From Equations (42), we can simplify Equation (43) as follows,

\[
\frac{PMT_j}{(1+i)^j} + \frac{PMT_k}{(1+i)^k} = \frac{PMT_{j}^{alt}}{(1+i)^j} + \frac{PMT_{k}^{alt}}{(1+i)^k}
\]

(44)

Since the buyer chooses to take a loan only when \(r > i\), thus \(\frac{1+i}{1+r} < 1\). Given that \(PMT_{j}^{alt} < PMT_j\) and \(PMT_{k}^{alt} > PMT_k\) \((k > j)\), and with Equation (44), we have

\[
\frac{PMT_j}{(1+i)^j} + \frac{PMT_k}{(1+i)^k} \times \left[ 1 + \frac{1}{1+r} \right]^{k-j} \quad \frac{PMT_{j}^{alt}}{(1+i)^j} + \frac{PMT_{k}^{alt}}{(1+i)^k} \times \left[ 1 + \frac{1}{1+r} \right]^{k-j}
\]

(45)

Multiplying both sides of Equation (45) by \(\left[ 1 + \frac{1}{1+r} \right]^j\) yields,

\[
\frac{PMT_j}{(1+r)^j} + \frac{PMT_k}{(1+r)^k} \quad \frac{PMT_{j}^{alt}}{(1+r)^j} + \frac{PMT_{k}^{alt}}{(1+r)^k}
\]

(46)

Thus we have,

\[
\sum_{t=1}^{T} \frac{PMT_t}{(1+r)^t} > \sum_{t=1}^{T} \frac{PMT_{t}^{alt}}{(1+r)^t}
\]

(47)

The total gain from trade is the sum of buyer’s gain and seller’s gain, which can be expressed as follows,

\[
\Pi^{max} = \left[ E[V_{max}^\ast] - E - \sum_{t=1}^{T} \frac{PMT_t}{(1+r)^t} \right] + (P - R)
\]

(48)

If the buyer takes the alternative mortgage, Equation (48) becomes,
\[ \Pi^{\text{max, alt}} = \left[ E[V^{\text{max}} | N^*] - E - \sum_{i=1}^{T} \frac{\text{PMT}_{i}^{\text{alt}}}{(1 + r)^i} \right] + (P - R) \] (49)

The first term in the bracket in Equations (48) and (49) is the buyer’s gain from trade. From Equations (47)-(49), we can conclude that the buyer’s gain increases due to delaying mortgage payments, which results in an increase in the total gain from trade,

\[ \Pi^{\text{max, alt}} > \Pi^{\text{max}} \] (50)

Therefore, delaying mortgage payments increase both the buyer’s gain and the total gain from trade, which results in higher transaction prices.

**End of the proof**

During the 2004-2006 housing boom, the U.S. home prices had a double-digit increase annually, which resulted in affordability issues in many areas across the country. Because interest-only (IO) mortgages allow borrowers to pay only interest portion, it became very popular during the housing boom. Compared with FRM loans, IO mortgages are a good example of delaying mortgage payments, and consequently it will lead to not only higher transaction price but also a larger LTV bias.

**Theorem 5.** The LTV bias of IO mortgages can be calculated as follows,

\[ \text{LTV}^{\text{bias}} = \text{LTV}^{\text{price-based}} \left( 1 - \frac{1 - \sum_{i=1}^{N} \frac{1}{(1 + r)^i} \text{LTV}^{\text{price-based}}}{1 - \frac{1 - \theta_{B}^{\text{Bgn}} \theta_{B}^{\text{Bgn}} (1 - \sum_{n=1}^{N} \frac{1}{(1 + r)^n} \text{LTV}^{\text{price-based}})}{1 - \theta_{B}^{\text{Bgn}} \theta_{B}^{\text{Bgn}} (1 - \sum_{n=1}^{N} \frac{1}{(1 + r)^n} \text{LTV}^{\text{price-based}})}} \right) \] (51)

**Proof:**

With IO mortgage, the present value of its payments in Equation (9) can be rewritten as follows,
\[ PV_B = (P - E) \left[ \sum_{n=1}^{N} i \left( \frac{1}{(1+r)^n} \right) + \frac{1}{(1+r)^N} \right] \] (52).

We can express \( \frac{\Omega_r}{\Omega_i} \) in Equation (9) as follows,

\[ \frac{\Omega_r}{\Omega_i} = \sum_{n=1}^{N} \frac{i}{(1+r)^n} + \frac{1}{(1+r)^N} \] (53).

Similarly, we can obtain the LTV bias of IO mortgages by using Equation (38) with replacing \( \frac{\Omega_r}{\Omega_i} \) in Equation (53) as follows,

\[ \text{LTV}^{\text{bias}} = \text{LTV}^{\text{commonly-used}} \left( 1 - \frac{1}{1 - \frac{\theta_{B}^{Bgn}}{1 - \theta_{S}^{Bgn} \theta_{B}^{Bgn}} \left( 1 - \sum_{n=1}^{N} i \frac{1}{(1+r)^n} - \frac{1}{(1+r)^N} \right) \text{LTV}^{\text{commonly-used}}} \right) \]

End of the proof

4. Simulations

To gauge the economic significance of the bias contained in price-based LTV ratios, we simulate \( \text{LTV}^{\text{bias}} \) under a variety of scenarios. We start with a base case with two assumptions. First, we assume \( \text{LTV}^{\text{price-based}} = 80\% \). Second, we assume that the buyer and the seller possess an equal level of bargaining power by evenly splitting the gain from trade. From Equations (16) and (17), we have \( \frac{1 - \theta_{B}^{Bgn}}{1 - \theta_{S}^{Bgn} \theta_{B}^{Bgn}} = 0.5 \). We allow the buyer’s discount rate \( r \) and the mortgage rate \( i \) to vary between 3\% to 10\% (with 1 percentage point increment), generating 36 different combinations of \( i \) and \( r \). For each
A combination of \( i \) and \( r \), we simulate \( LTV^{bias} \) with a 30-year and with a 15-year FRM. Results are presented in Table 1.

### Table 1: Simulation Results of \( LTV^{bias} \) (Base Case Scenario)

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Several important observations can be made. First, when \( r \) equals \( i \), \( LTV^{bias} = 0 \). In this case, financing makes no contribution to the total gain from trade, and a buyer is indifferent between getting a mortgage and a cash purchase. As a result, the property transacts at its collateral value, and the price-based LTV ratio is unbiased. When \( r \) is
strictly greater $i$, financing enhances the total gain from trade, and transaction price exceeds the property’s collateral value.\textsuperscript{15} Consequently, price-based LTV ratio understates financial leverage. As shown in Table 1, when $r > i$, $LTV^{\text{bias}}$ is strictly negative, which is consistent with the prediction of our theoretical model. This finding is also in line with the results of many empirical studies that properties with mortgage financing sell higher prices than those with cash transactions.\textsuperscript{16} Our simulation enables an examination of the magnitude of $LTV^{\text{bias}}$. For example, when $r=4\%$ and $i=3\%$, $LTV^{\text{bias}}$ is -3.92 percentage points. To put this in perspective, at a price-based LTV of 80\%, our simulation reveals the true financial leverage, $LTV^{\text{true}}$, is 83.92\%. Furthermore, $LTV^{\text{bias}}$ exacerbates as the gap between $r$ and $i$ grows wider. With a $r=10\%$ and $i=3\%$, $LTV^{\text{bias}}$ is a whopping -20.99 percentage points, which suggests an 100.99\% true financial leverage. A mortgage originated at an 80\% LTV is, in fact, already underwater at loan origination. We can also see that the same gap between $r$ and $i$ produces a greater $LTV^{\text{bias}}$ in a low-interest environment. For example, $LTV^{\text{bias}}$ is -3.92 percentage points when $r=4\%$ and $i=3\%$. In comparison, it is -2.75 percentage points, when $r=10\%$ and $i=9\%$. We ran a similar simulation with a 15-year FRM. Results are presented in the bottom panel of Table 1. First, all patterns previously identified from the simulation results of a 30-year FRM continue to exhibit here. It is also noticeable that in comparison to a 30-year FRM, $LTV^{\text{bias}}$ is, \textit{ceteris paribus}, reduced by the shorter term. This is consistent with our prediction that the ability to postpone repayments leads to a larger $LTV^{\text{bias}}$.

\textsuperscript{15} When $r$ is strictly less than $i$, a buyer will not have a mortgage because he is worse off by taking it. As a result, the buyer will choose cash transaction. Consequently, the property will transact at its collateral value, and the price-based LTV ratio is unbiased.

\textsuperscript{16} For example, Lusht and Hansz (1994) find a 16.5\% discount for cash-only transactions in Lehigh County, Pennsylvania. More recently, Aroul and Hansz (2011) and Jauregui et al (2017) find 13.5\% and 9\% discount for cash transactions in Clovis, California and Franklin County, Ohio, respectively.
In addition to the base-case scenario, we further simulate a 30-year FRM with a higher financial leverage greater than 80%. Specifically, we set \( \text{LTV}^{\text{price-based}} \) to be 90% and 95%. Results are presented in Table 2.

### Table 2: Simulation Results of \( \text{LTV}^{\text{bias}} \) (High Financial Leverage)

#### 30-Year Mortgage (LTV = 90%, \( \frac{1 - \theta_{\text{Bgn}}}{1 - \theta_{\text{Bgn}}^* \theta_{\text{Bgn}}^*} = 0.5 \))

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#### 30-Year Mortgage (LTV = 95%, \( \frac{1 - \theta_{\text{Bgn}}}{1 - \theta_{\text{Bgn}}^* \theta_{\text{Bgn}}^*} = 0.5 \))

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Table 2 indicates that increasing financial leverage drastically exacerbates \( \text{LTV}^{\text{bias}} \). For example, with \( r=4\% \) and an \( i=3\% \), \( \text{LTV}^{\text{bias}} \) changes from -3.92 percentage points to -5.00 percentage points when \( \text{LTV}^{\text{price-based}} \) increases from 80% to 90%. \( \text{LTV}^{\text{bias}} \)
becomes -5.59 percentage points when $\text{LTV}^{\text{price-based}}$ is set to be 95%. When high financial leverage is accompanied by a wide gap between $r$ and $i$, the bias contained in the price-based LTV ratio is disastrous. At a $\text{LTV}^{\text{price-based}}$ of 95%, a 10% of $r$ and a 3% of $i$ would produce a $\text{LTV}^{\text{bias}}$ of -31.13 percentage points. In other words, the loan amount of a mortgage originated at a 95% LTV is in fact 26.13% greater than the collateral value of the property!

Next, we explore the influence of bargaining power on $\text{LTV}^{\text{bias}}$. So far, we assumed the buyer and the seller possess equal bargaining powers. However, in a hot market, increased demand for housing may put the seller in an advantageous bargaining position and allow her to get a larger share of the pie. To simulate such a market environment, we first increase the seller’s share of the total gain from trade, $\frac{1-\theta_B^{\text{Bgn}}}{1-\theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}}$, from 0.5 to 0.8. Results are reported in Table 3.

Observing from the top panel of Table 3, enhanced seller’s bargaining power exacerbates $\text{LTV}^{\text{bias}}$. For example, with $r=4\%$ and $i=3\%$, $\text{LTV}^{\text{bias}}$ goes from -3.92 percentage points in the base-case scenario to -6.47 percentage points. As shown in the bottom panel of Table 3, $\text{LTV}^{\text{price-based}}$ understates default risk even more when the seller possess all the bargaining power (e.g. $\frac{1-\theta_B^{\text{Bgn}}}{1-\theta_S^{\text{Bgn}} \theta_B^{\text{Bgn}}} = 1$). $\text{LTV}^{\text{bias}}$ becomes -8.25 percentage points when $r = 4\%$ and $i = 3\%$. 
Table 3: Simulation Results of $LTV^{bias}$ (High Seller Bargaining Power)

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Finally, we simulate $LTV^{bias}$ with an interest-only loan. To establish comparability, we adopt the same set of parameters used for our base-case scenario, in which a fully-amortizing FRM with constant payments is analyzed. Simulated values of $LTV^{bias}$ are presented in Table 4. It is clear that the alteration of amortization structure from fully-amortizing to interest-only substantially exacerbates $LTV^{bias}$. For example, with $r=4\%$ and $i=3\%$, $LTV^{bias}$ changes from -3.92 percentage points in the base case to -6.00 percentage points. Examining other combinations of $r$ and $i$, we observe $LTV^{bias}$ is
worsen across the board under the interest-only loan. The bottom panel of Table 4 presents simulated $LTV^{bias}$ resulted from a 15-year interest-only loan. Results are qualitatively similar. This set of results is consistent with our model’s prediction that postponement of repayments leads to a larger $LTV^{bias}$. Relative to a fully-amortizing FRM, an interest-only loan allows a borrower to delay repayments.

**Table 4: Simulation Results of $LTV^{bias}$ (Interest-Only Mortgage)**

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15-Year Interest-Only Mortgage (LTV = 80%, $\frac{1-\theta_B^{gen}}{1-\theta_S^{gen} \theta_B^{gen}} = 0.5$)

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5. Conclusion

Real estate prices are often established through bargaining and financed with mortgage debt. We study the effect of bargaining and financing on prices. In this paper, we explore the price formation of a mortgage-financed property in a bilateral bargaining game. We find that when the price of a mortgage-financed property is formed through bargaining, the resulted transaction price reflects not only the value of the property, it also captures the value created through financing. As a result, mortgage-financed properties can trade at prices that are well above their collateral values, and the loan-to-value (LTV) ratio contains a bias that understates credit risk. This bias is exacerbated when a mortgage is originated with a longer term, at a higher LTV ratio, or when the seller possesses stronger bargaining power relative to the buyer. Furthermore, this bias is larger under aggressive lending products, such as interest-only loans and mortgages that allow negative amortization. Our simulation results suggest that many mortgages originated at the peak of the housing bubble are likely “under water” at origination. In particular, the loan amount of a 30-year mortgage at a 95% LTV can be 26.13% greater than the collateral value of the property, suggesting the mortgage is already deep “under water” at origination. The bias of the LTV ratio also exists in many other collateralized debt (e.g. commercial mortgages, car loans).

Our findings cast doubt on the widely adopted practice of using transaction price to estimate collateral value and call into questions underwriting and risk control practices of mortgage industries and other collateralized debts. For example, in the United States, an 80% LTV is the cutoff for determining whether or not private mortgage insurance is required on a conventional mortgage. This 80% cut-off has been consistently applied by
Fannie Mae and Freddie Mac for many years. Our findings suggest that this lack of variation on the cutoff is likely to be suboptimal. Since 2000, both U.S. mortgage market and housing market have undergone some dramatic changes. For example, mortgage rates declined continually and repeatedly hit historical lows. In additional, U.S. housing market experienced a full boom-and-bust cycle. Our analysis suggests that with changing market conditions, it is unlikely an 80% LTV represents the same level of credit risk at different times. An optimal credit risk control strategy should involve lowering the LTV cutoff during periods when credit supply are increasing, demand for housing is high, and aggressive lending products are widely used.
Reference


Appendix I

Suppose that buyers’ valuations are uniformly distributed over \([V, \overline{V}]\), hence their probability density function is

\[
f(V_j) = \begin{cases} \frac{1}{(\overline{V} - V_j)}, & V_j \in [V, \overline{V}] \\ 0, & \text{otherwise} \end{cases}
\]  

(A1).

The cumulative distribution function of buyers’ valuation is

\[
F(V_j) = \begin{cases} 1, & V_j > \overline{P} \\ \frac{V_j - V}{(\overline{V} - V_j)}, & V_j \in [V, \overline{V}] \\ 0, & \text{otherwise} \end{cases}
\]  

(A2).

Given that \(V_{\text{max}}\) is the highest valuation among all available buyers, the density function of \(V_{\text{max}}\) is

\[
g_{V_{\text{max}}}(V) = N \zeta F(V)^{N-1} f(V)
\]  

(A3).

Therefore, we have

\[
E[V_{\text{max}}] = \int_{V} V \zeta N \frac{1}{\overline{V} - V} \left(\frac{V - V_j}{\overline{V} - V_j}\right)^{N-1} dV = \frac{\zeta N \overline{V} + V}{\zeta N + 1}
\]  

(A4).